STRENGTH OF MATERIALS

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STRENGTH OF MATERIALS

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PREFACE TO THE FIFTH EDITION

In this revision a conscious attempt has been made to emphasize the fundamentals of the theory and accent the process of analysis in the application of mathematics to strength of materials. The objective has been to inculcate sound methods by utilizing fundamental ideas, physical constants, and reasonable stresses with a minimum of memory formulas. A comprehensive grasp and working knowledge of the subject can be obtained only by a thoughtful application of principles. While this may require more time in the early development of the student, it enhances his growth later.

Several chapters have been rewritten. More emphasis has been placed on the area-moment method. The chapter on deflection by double integration has been retained, but the direction has been focused toward a logical development of the topic rather than the mechanical manipulation of equations. Integration between limits has been omitted. By the early introduction of Mohr's circle for the solution of simple stresses, it is hoped that the student will gain confidence before he is confronted with combined stress. Considerable new material has been added.

Some changes have been made in the order of topics. Shearing stresses in beams now immediately follows bending stresses. Special beam topics, including reinforced-concrete beams, have been delayed until after combined stress. References to both steel and aluminum rolled sections are made in the text and in some problems which require the use of the AISC Manual of Steel Construction and the Alcoa Structural Handbook. Many new problems have been added, making the total well over a thousand. The more difficult problems have been placed in miscellaneous lists at the ends of the chapters.

The writer acknowledges his indebtedness to Prof. J. E. Boyd, who pioneered this work and furnished much enthusiasm. His colleagues, Profs. P. W. Ott, R. W. Powell, E. C. Clark, and C. T. West have been very generous with their ideas and inspiration. Prof. M. G. Fontana of the Department of Metallurgy and Dr. John Zambrow of the Engineering Experiment Station furnished new data. Richard I. Hang of the Department of Engineering Drawing made the drawings, which uphold the tradition of the previous editions.

S. B. Folk

Columbus, Ohio
March, 1950

PREFACE TO THE FIRST EDITION

This book is intended to give the student a grasp of the physical and mathematical ideas underlying the Mechanics of Materials, together with enough of the experimental facts and simple applications to sustain his interest, fix his theory, and prepare him for the technical subjects as given in works on Machine Design, Reinforced Concrete, or Stresses in Structures.

It is assumed that the reader has completed the Integral Calculus, and has taken a course in Theoretical Mechanics which includes statics and the moment of inertia of plane areas. Chapters XVI and XVII give a brief discussion of center of gravity and moment of inertia. Students who have not mastered these subjects should study these chapters before taking up Chapter V (preferably before beginning Chapter I).

The problems, which are given with nearly every article, form an essential part of the development of the subject. They were prepared with the twofold object of fixing the theory and enabling the student to discover for himself important facts and applications. The first problems of each set usually require the use of but one new principle—the one given in the text which immediately precedes; the later problems aim to combine this principle with others previously studied and with the fundamental operations of Mathematics and Mechanics. The constants given in the data or derived from the results of the problems fall within the range of the figures obtained from actual tests of materials. Many of the problems are taken directly from such measurements. Some of them are from tests made by the author or his colleagues at the Ohio State University; others are from bulletins of the University of Illinois Engineering Experiment Station, from "Test of Metals" at the Watertown Arsenal, and from the Transactions of the American Society of Civil Engineers.

This book is designed for use with "Cambria Steel," to which references are made by title instead of by page, so that they are adapted to any edition of the handbook.

The author acknowledges his indebtedness for suggestions and criticisms to Professors C. T. Morris, E. F. Coddington, Robert Meiklejohn, K. D. Swartzel, and many others of the Faculty of the

College of Engineering; and to Professor Horace Judd of the Department of Mechanical Engineering for the material for several of the half-tones. He also expresses his obligations to the books which have helped to mold his ideas of the subject,—Johnson's "Materials of Construction," Ewing's "Strength of Materials," and especially the textbooks which he has used with his classes,—Merriman's "Mechanics of Materials," Heller's "Stresses in Structures," and Goodman's "Mechanics Applied to Engineering."

The symbols used in the mathematical expressions are much the same as in Heller's "Stresses in Structures."

J. E. B.

COLUMBUS, OHIO
November 6, 1911

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NOTATION

The symbols which are frequently used in this book are

a = radius of circle; distance of concentrated load from left end of span.

A = area; area of cross section; area enclosed by center line of thin hollow tube.

A' = some special area; part of the area of a cross section.

b = breadth; breadth of rectangular section; base of triangle; distance of concentrated load from right end of span.

B = some special value of b, usually a maximum; maximum breadth of a constant-strength beam.

c = distance from neutral axis to extreme outer fiber; distance of center of a circular curve beam from center of curvature; distance in figure.

C = distance from center of curvature of trapezoidal curved beam to intersection of sides.

C_1, C_2, C_3 = integration constants.

d = depth; depth of rectangular section; diameter; distance between parallel axes.

D = some special depth; diameter of boiler; maximum depth of a constant-strength beam.

e = eccentricity of a load on a column; distance in figure.

E = modulus of elasticity.

E_c = modulus of elasticity in compression; modulus of elasticity of concrete.

E_s = tension modulus of elasticity of steel reinforcement.

E_t = modulus of elasticity in tension.

E_w = working modulus of elasticity.

G = modulus of rigidity, or modulus of elasticity in shear.

h = height; distance from vertex to base of triangle.

hp = horsepower.

H = product of inertia; distance of larger base of trapezoidal curved beam from intersection of nonparallel sides.

H_c = product of inertia with respect to axes which intersect at center of gravity.

H' = product of inertia for inclined axes.

I = moment of inertia.

I_c = moment of inertia with respect to axis through center of gravity.

I_m = maximum moment of inertia of beam of variable section.

I_x = moment of inertia with respect to X axis.

I_y = moment of inertia with respect to Y axis.

j = ratio of moment arm to total depth of a reinforced-concrete beam.

J = polar moment of inertia.
k = a constant coefficient; a ratio less than unity.
K = modulus of volume elasticity.
l = length; length of beam between supports; length of column between points of inflection.
L = length; total length of column.
m = mass of particle; slope of tangent at support; a ratio.
M = moment; mass.
M_o = moment at origin of coordinates.
M_a, M_b, M_c = moment over three consecutive supports.
M_1, M_2, M_3, etc. = moment over first, second, third, etc., supports.
M_p = moment caused by a dummy load P.
M_q = moment caused by actual loads.
M_t = moment caused by a dummy couple.
n = ratio; number of turns in a helical spring.
N = normal force at surface; number of revolutions per minute.
p = pitch of rivets; slope of tangent; ratio of steel area to concrete area; maximum normal stress.
P = concentrated load or force; internal pressure in thin cylinders.
q = coefficient in Rankine's formula; minimum normal stress; shear flow.
Q = concentrated load or force.
r = distance from origin; radius of gyration (in column formulas); radius.
R = reaction at support; resultant force; radius; radius of coil.
R_1 = reaction at left support; radius of inside surface of curved beam or hook.
R_2 = reaction at second support; radius of outside surface of curved beam or hook.
R_0 = radius of neutral surface of curved beam or hook.
$\bar{R}$ = radius of curved beam to center of gravity of section.
s = unit stress.
s_t, s_s, s_c = unit tensile, shearing, and compressive stress.
s_u = ultimate unit stress.
s_w = allowable unit stress.
s' = unit stress resulting from combined shear and tension or compression.
S = unit stress in extreme fibers.
S_1 = unit stress at concave surface of curved beam.
S_2 = unit stress at convex surface of curved beam.
S_s = unit shearing stress at surface of shaft.
t = thickness.
T = torque; tension.
u = work, or energy per unit volume; bond stress in concrete beams.
U = work.
U_p = modulus of resilience.
v = distance from neutral axis; shearing stress in concrete beams.
$\bar{v}$ = distance from neutral axis to center of gravity of A'.

V = total vertical shear.
V_{ab} = total shear near support A in span joining A to B.
w = distributed load per unit of length.
W = total load uniformly distributed.
$\bar{x}, \bar{y}, \bar{z}$ = coordinates of center of gravity.
y = deflection in a beam or column.
y_a = deflection at A caused by unit load at A.
y_{ba} = deflection at A caused by unit load at B.
y_p = deflection under load P caused by this load.
y_{max} = maximum deflection of a beam or column.
Z = section modulus.
α = coefficient of thermal expansion; shear deflection factor.
$\alpha, \beta, \theta, \phi$ = angles in figure.
ϵ = unit linear deformation.
γ = unit shearing deformation.
μ = Poisson's ratio; coefficient of friction.
ρ = density; radius of curvature.
θ_1, θ_2 = slope at 1 and 2 from left to right.
θ_{21} = slope at 2 from right to left.
τ = maximum resultant shearing stress.
Σ_0 = perimeter of rods in reinforced concrete.

CHAPTER 1

STRESSES

1. Strength of Materials. That branch of mechanics which treats of the changes in form and dimensions of elastic solids and the forces which cause these changes is called *the mechanics of materials.* When the physical constants and the results of experimental tests upon the materials of construction are included with the theoretical discussion of the ideal elastic solid, the entire subject is called *the strength of materials* or *the resistance of materials.*

2. Tension. Figure 1 shows a rubber band which is suspended from a horizontal bar and carries a hook at the lower end. When a small weight is hung on the hook, the rubber band is stretched; its length is increased by an amount a, while its cross section is reduced. When a second weight is added, there is an additional elongation b. If the weights are equal, the elongation a caused by the first weight is equal to the elongation b caused by the second weight. When the weights are removed, the rubber band returns to its original length and cross section.

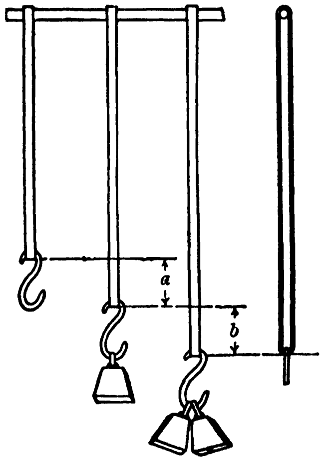

FIG. 1. Rubber bands in tension.

If steel, iron, wood, concrete, stone, or other solid material is used instead of rubber, the results are similar. There is this apparent difference: while the rubber may be stretched to twice or three times its original length and still return to its original size and shape after the load is removed, one of the other materials may be stretched only a very small amount (usually less than 0.002 of its length), without receiving a permanent change in its dimensions. Again, the force required to produce a relatively small increase in the length of a rod of wood or steel, for instance, is many times greater than that necessary to *double* the length of a soft rubber band of equal cross section. These differences between the behavior of soft rubber and other solid materials are differences of degree and not of kind. Essentially they are alike.

The rubber bands shown in Fig. 1 are subjected to the action of two forces: the force of the weights pulling downward, and the reaction of the support pulling upward. The bands are in *tension*. A body is said to be in tension when it is subjected to two sets of forces whose resultants are in the same straight line, opposite in direction, and directed *away* from each other.

3. Compression. When a body is subjected to two sets of forces whose resultants are in the same straight line, opposite in direction, and directed *toward* each other, it is said to be in *compression*. In Fig. 2, the block B is in compression under the action of the 50 pounds pushing down and the reaction of the support pushing up. The effect of compression upon a body is to shorten it in the line of the forces and increase its dimensions in the plane perpendicular to this line.

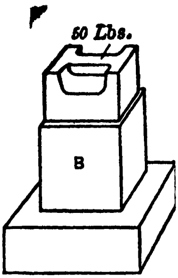

FIG. 2. Compression.

Tension and compression may be represented as in Fig. 3, in which the arrows represent the forces, and the small rectangles represent the bodies, or portions of a body, upon which the forces act. The rectangles are often omitted; a pair of arrows with their heads together indicates compression, and a pair with their heads in the opposite sense indicates tension.

4. Force. The force exerted by one body on another at their surface of contact produces a stress in the bodies. In Fig. 2 the total force is 50 pounds. The *stress* produced is compressive in block B. The support pushes up against the body with an equal force. The total load on a body will always be called the *force*, or *load*.

Figure 4 represents a bar subjected to a horizontal pull of P pounds. If the bar is supposed to be cut by an *imaginary plane* at C, the portion A to the left of this *plane section* is in equilibrium under the action of the external pull P_1 toward the left and an equal opposite pull P_3 at the section C. This force P_3 across the section is the pull exerted by the right portion B upon the left portion A. In like manner, the right portion B is in equilibrium under the external pull P_2 at the right end of the bar and the internal P_4, equal and opposite to P_3, exerted by the left portion A upon the right portion B across the section, as shown separately in Fig. 4,II.

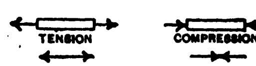

FIG. 3.

Figure 4,II is a free-body diagram for body B. Equilibrium of a free body is maintained by the action of *total forces* on the body. These forces may be internal or external.

5. Unit Stress; Intensity of Stress. The average *unit compressive* or *tensile stress* at any section of a body is calculated by dividing the total force by the area of the cross section at right angles to the force. If a vertical force P is applied to the cylinder C of Fig. 5 by means of the plate B and the reaction of the support D, the unit stress at any section is given by the equation

$$s = \frac{P}{A} \qquad \text{Formula I[1]}$$

in which s is the *unit stress*, P is the *external* force, and A is the area of cross section perpendicular to the direction of the stress. *Unit stress*

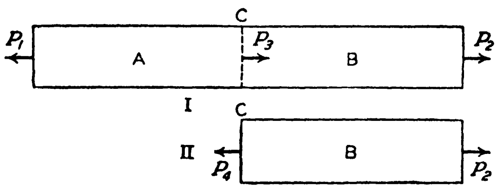

FIG. 4. Stress at section.

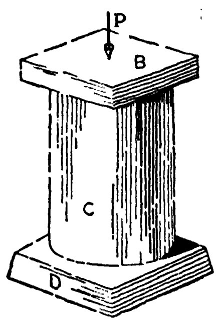

FIG. 5. Area under stress.

frequently is called *intensity* of *stress*. In American engineering practice, unit stresses generally are given in pounds per square inch or kips per square inch. (One *kip* or kilo pound is 1,000 pounds.) Frequently compressive stresses in large masonry structures are expressed in tons per square foot. It is the common practice to give bearing pressure of masonry on soils in this way. British engineers employ long tons per square inch as well as pounds per square inch to express the intensity of stress in steel and similar materials. Continental[2] engineers, of course, use kilograms per square centimeter. Physicists prefer dynes per square centimeter or dynes per square millimeter. Stress in pounds per square inch may be written psi.

In elementary mechanics the tensile or compressive stress exerted by a bar is usually assumed to lie in the axis of the member. In reality each longitudinal element exerts its portion of the stress. The force

[1] Important formulas, which should be understood and memorized, are designated by Roman numerals in this book.

[2] They sometimes use atmospheres. One atmosphere equals 14.7 pounds per square inch, or 1.033 kilograms per square centimeter.

assumed to act along the axis is the resultant of the forces exerted by all the elements. The unit stress obtained by dividing the total applied force by the area of the cross section is the *average unit stress* in the member.

Figure 6,I shows a bar under tensile stress which is uniform in all parts of the section. The arrows which represent the stress of different elements are all of equal length. Figure 6,II shows a bar under uniform compressive stress. Figure 6,III shows compressive stress which increases uniformly from left to right.

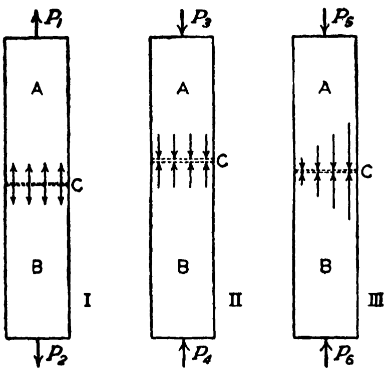

FIG. 6. Representation of stress.

When the stress is uniform, the resultant stress passes through the center of gravity of each cross section, which corresponds to the center of gravity of a short piece of uniform length cut from the bar. When the stress is not uniform, the location of the resultant may be found by calculating the sum of the moments with respect to some parallel plane of the force on each element of area and dividing this moment by the sum of the forces. In other words, the resultant passes through the center of gravity of a solid whose base is the section of the bar and whose altitude at any point is proportional to the unit stress at that point.

Problems

(Find the dimensions of rolled shapes in a steel or aluminum handbook.)

5-1. The cylinder of Fig. 5 is 2 in. in diameter and 5 in. long. Find the unit stress when a vertical load of 14,000 lb is applied by means of the plate *B*.
Ans. 4,456 psi.

5-2. A piece 6 in. long is cut from a 5- by 3- by ½-in. angle section by planes perpendicular to its length. The piece stands vertical and a load of 30,000 lb is applied at the top by means of a 5- by 3- by 1-in. steel plate. Find the unit stress in the angle. *Ans.* s = 8,000 psi.

5-3. Two edges of the plate in Prob. 5-2 lie in the planes of the back of the legs of the angle section. The load is applied to the plate by means of a steel ball. Where must this ball be placed in order that the unit stress in the angle may be uniform?
Ans. 1.75 in. from one 3-in. edge, and 0.75 in. from one 5-in. edge.

5-4. A 10-in. 25.4-lb standard I beam 12 in. long stands on end and carries a total vertical load of 15,000 lb on top. Find the average unit stress.
Ans. 2,030 psi.

5-5. The beam in Prob. 5-4 rests on one flange and the load is applied on the top flange. Find the maximun unit stress and indicate the cross section where it occurs. *Ans.* 4,030 psi.

5-6. A 12-in. length of 6-in. standard pipe is placed between two steel plates and subjected to a force of 30,000 lb. Find the unit compressive stress in the cross section. *Ans.* 5,380 psi.

5-7. A 4- by 3-in. 2.84-lb aluminum tee section 12 in. long stands on end and carries an 8,000-lb load on top. Find the unit stress in a cross section. *Ans.* 3,420 psi.

5-8. A block in the form of a frustum of a pyramid is 2 in. square at the top, 3 in. square at the bottom, and 8 in. high. Find the unit stress 2 in. from the bottom and 4 in. from the bottom when a load of 7,200 lb is placed on the top. *Ans.* 952.1 psi; 1,152 psi.

5-9. In a short block 2 in. square, the unit stress increases uniformly from 100 psi in the left face to 700 psi in the right face. Find the total load. *Ans.* $P = 1{,}600$ lb.

5-10. In Prob. 5-9, find the location of the resultant force. Represent the stress in the front face by a trapezoid 100 units high on the left and 700 units high on the right. Find the center of gravity of the trapezoidal wedge which represents the force by combining the moment and area of two triangles, or the moment and area of a triangle and a rectangle. *Ans.* 1.25 in. from the left face; 1 in. from the front face.

5-11. A short block of triangular section has two faces each 13 in. wide, and one face 10 in. wide. The block is subjected to compression parallel to its length which causes the unit stress to increase uniformly from 100 psi at the intersection of the 13-in. faces to 700 psi in the 10-in. face. Find the total load by integration. Show that this load equals the area of the section multiplied by the unit stress at the center of gravity of the cross section. *Ans.* $P = 30{,}000$ lb.

5-12. By integration of moments, find the line of action of the resultant force of Prob. 5-11. *Ans.* 8.8 in. from the intersection of the 13-in. faces.

6. Working Stress; Allowable Unit Stress. Working stresses are the unit stresses to which the materials of a machine or structure are subjected. The *allowable unit stress* for a given material is the maximum unit stress which, in the judgment of some competent and official authority, should be applied to this material. For instance, the specifications of the American Institute of Steel Construction give 20,000 pounds per square inch as the unit tensile stress for structural steel. For the compressive stress in relatively short blocks of select-grade white oak in situations which are always dry, the American Society for Testing Materials specifies 1,000 pounds per square inch parallel to the grain. The Joint Committee of Concrete and Reinforced Concrete[1] gives 25 per cent of the compressive strength at 28 days as the allowable compressive stress of concrete.

[1] This committee is made up of representatives from the American Society

Table 1 gives a few allowable stresses in tension and compression.

TABLE 1. ALLOWABLE UNIT STRESS
(This table should be memorized.)

Material	Tension, psi	Compression, psi	
Structural steel	20,000	20,000	
Cast steel	16,000	16,000	
Wrought iron	12,000	12,000	
Cast iron	3,000	15,000	
Nickel steel	25,000	25,000	
Bolts on nominal area at root of thread	20,000		
Butt welds, section through throat	20,000		
Aluminum alloy 17S-T and 24S-T	15,000	15,000	
Portland cement concrete	.....	600	
		With grain	Across grain
Common-grade timber in dry location:			
Douglas fir, coast region	......	880	325
Southern yellow pine	......	880	325
White or red oak	......	800	500

A steel bar 1 *foot long and* 1 *square inch in cross section weighs* 3.4 *pounds.* For estimating purposes 1 cubic inch of steel weighs 0.283 pounds and 1 cubic inch of aluminum weighs 0.1 pound, although alloys will vary considerably from these figures.

Problems

(Use the data of Table 1 *unless otherwise specified.)*

6-1. Find the total allowable load, in compression parallel to the grain, which may be applied to a 4- by 6-in. short block of southern yellow pine.

Ans. 21,120 lb.

6-2. What must be the dimensions of a cubical block of white oak which supports a load of 50,000 lb? (Two solutions.)

6-3. An I bar of structural steel, 1 in. thick, exerts a pull of 60,000 lb. What is its minimum width?

6-4. A piece of 6-in. wrought-iron water pipe is 2 ft long and 6⅝ in. in outside diameter. What is the allowable load on the pipe standing on end?

Ans. 66,970 lb.

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of Civil Engineers, the American Society for Testing Materials, the American Railway Engineering Association, the American Concrete Institute, and the Portland Cement Association.

6-5. A yellow-pine beam 8 in. wide rests on the end of the pipe of Prob. 6-4. A steel plate, 10 in. square, transmits the load from the beam to the pipe. Find the allowable load. *Ans.* 26,000 lb.

6-6. A 1-in. steel bolt supports a load by means of a square standard nut. What is the allowable load? (Use AISC handbook.) *Ans.* 11,020 lb.

6-7. In Prob. 6-6 what is the bearing stress on the nut when the bolt carries its allowable load? *Ans.* 7,720 lb.

6-8. The bolt of Prob. 6-6 runs vertically through an oak beam. Find the diameter of the washer required.

6-9. In Fig. 7 the pin-connected steel truss is hinged at J and held by a horizontal force at A. Find the diameter of the round rod AB. *Ans.* 1.67 in.

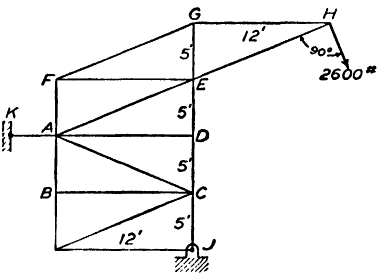


FIG. 7.

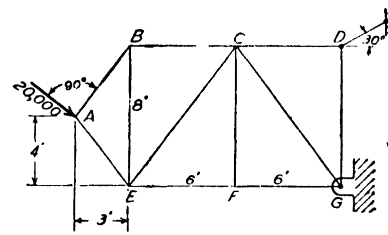


FIG. 8.

6-10. The pin-connected aluminum truss shown in Fig. 8 is hinged to the wall at G and held by a force at D. Find the diameter of a round rod to be used at CD. *Ans.* 1.28 in.

6-11. What is the allowable load in tension on a steel rod which is 5 ft 6 in. long and weighs 70 lb?

7. Deformation; Unit Deformation. The changes in dimensions which occur when forces are applied to a body are called *deformations.* In Fig. 1, the increase in length a, which takes place when the first load is applied, is the deformation caused by that load; the increase b is the deformation caused by the second load; and $a + b$ is the deformation caused by the two loads. The deformation produced by a *tensile* force or *pull* is an *elongation.* The deformation produced by a *compressive* force or *push* is a *compression.* Compression is negative elongation. A deformation which remains after the force is removed is called a *set.*

Unit deformation in a body is the deformation per unit length. In a bar of uniform cross section, the unit deformation is calculated by dividing the total deformation of a given portion of the bar by the original length of the portion. In Fig. 1, the length a divided by the original length of the band is the unit deformation caused by the first load. Unit deformation is frequently called *relative deformation.*

In algebraic equations many authors represent unit deformation by the letter ϵ (epsilon).

Deformation is frequently called *strain*. The word *strain* was formerly used as a synonym for *stress* and is still sometimes heard in that sense. The general practice of technical literature, however, is now to use *strain* to mean *deformation*. When employed in this book, it will always have that meaning. Total deformation in a length L sometimes is represented by e. Unit deformation is then

$$\epsilon = \frac{e}{L} \qquad \text{Formula II}$$

Problems

7-1. When a steel bar is subjected to a tensile stress, a portion, originally 8 in. long, is stretched 0.0052 in. Find the unit elongation. *Ans.* 0.00065

7-2. An oak post under compression is shortened 0.1476 in. in a length of 15 ft. Find the unit deformation. *Ans.* 0.00082

7-3. A ⅞-in. steel rod 20 in. long is subjected to a pull of 15,176 lb. A portion of the rod, originally 8 in. long, is stretched 0.0054 in. when the force is applied. Find the unit stress and the unit deformation.

7-4. The coefficient of expansion of steel is 0.000012 for 1°C. Find the unit deformation and the total deformation in a steel rod 15 ft long when the temperature changes from 50 to 20°C. Solve when the temperature changes from 14°F to 14°C. *Ans.* 0.000288; 0.05184 in.

8. Elastic Limit. When a force is applied to a solid body and then removed, the body returns to its original size and shape, provided the unit stress developed by the force has not exceeded a certain limit. If the stress has gone beyond this limit, the body does not return entirely to its original dimensions but retains some permanent deformation or *set*. The *unit stress* at this limit is called the *elastic limit* of the material. A soft-steel rod may be stretched 0.0054 inch in a gage length of 8 inches by a pull of 20,000 pounds per square inch. When this load is removed, the rod shortens to its original length. A pull of 30,000 pounds per square inch may stretch this rod 0.0081 inch, and the rod may return to its original length when the load is removed. A load of 32,000 pounds per square inch may stretch the rod 0.0200 inch. When this load is removed, the rod may have an elongation of 0.0110 inch. The rod shortens about 0.0090 inch, while the remaining elongation of 0.0110 inch persists as a permanent set. Evidently, the elastic limit is between 30,000 and 32,000 pounds per square inch.

It is difficult to determine the elastic limit with exactness. A test piece may appear to have no residual deformation when measured with the usual apparatus and still show some set when more delicate instru-

ments are employed. Time is a factor. If a load is applied for a considerable period, it causes somewhat greater deformation and considerably greater set than it would cause if the time of application were shorter. Some materials, such as steel, after having been subjected to comparatively large unit stress, frequently show a set of more or less temporary character. When the load is first removed, there is a residual deformation, which may partly or wholly vanish after some little interval.

9. Modulus of Elasticity. For all stresses below the elastic limit, the ratio of the unit stress to the unit deformation is *nearly* constant. The quotient obtained by dividing any given change of unit stress by the accompanying change in unit deformation is called the *modulus of elasticity* or *Young's modulus.*[1] Modulus of elasticity is represented in physical equations by the letter E. In algebraic language, the definition of the modulus of elasticity is

$$E = \frac{s}{\epsilon} \qquad \text{Formula III}$$

in which E is the modulus of elasticity, s represents a change in the unit stress, and ϵ is the change in unit deformation which accompanies this change of unit stress.

Problems

9-1. A 2- by 1.5-in. bar is tested in tension. When the load changed from 3,000 to 48,000 lb, the dial reading for a gage length of 8 in. changed from 0.00080 to 0.00492 in. Find the change in unit stress, the change in unit deformation, and the modulus of elasticity. *Ans.* E = 29,130,000 psi.

9-2. A steel rod 0.600 in. in diameter is stretched 0.00536 in. in a gage length of 8 in. when the load changed from 1,415 to 7,057 lb. Using the area to three significant figures, find the modulus of elasticity. *Ans.* E = 29,850,000 psi.

9-3. A timber piece 2 in. square is shortened 0.014 in. in a length of 20 in. Find the force required if the modulus of elasticity is 2,000,000 psi. How does the unit stress compare with the allowable compressive stress for southern pine?

9-4. A 10-in. 30-lb standard channel 10 ft long is subjected to a compressive load of 88,000 lb parallel to its length. How much is the channel shortened if E = 29,300,000? *Ans.* 0.0410 in.

9-5. A 4- by 3- by $\frac{5}{16}$-in. 2.84-lb aluminum tee 12 in. long shortened 0.000876 in. in an 8-in. gage length when the total load on the end increased from 484 to 6,292 lb. Find the modulus of elasticity. *Ans.* 10,960,000 psi.

9-6. In the pin-connected steel truss of Fig. 7, the member AE is a $\frac{1}{2}$-in. square bar. If the modulus of elasticity is 30,000,000 psi, find the total change in length which occurs when the 2,600-lb load is applied.

[1] Dr. Thomas Young (1773–1829), who was not an engineer, brought the idea to the attention of the English engineers.

9-7. The modulus of elasticity of the aluminum truss in Fig. 8 is 10,600,000 psi. The member CB is a rectangular bar 1 by ½-in. What is the approximate increase in length of CB?

9-8. In a tension test of cast iron at the Watertown Arsenal, an increase of unit stress from 1,000 to 6,000 psi produced an increase in length of 0.0034 in. in a gage length of 10 in. Find E for this cast iron.

9-9. A spruce stick 1.745 by 1.756 in., tested at the Bureau of Standards, was 25.25 in. in length. Deformations were measured on a 20-in. gage length. Some readings were

Total load, lb	*Average of two gages, in.*
1,224	0.00223
1,836	0.00424
4,284	0.01365
4,896	0.01620

Calculate the area to two decimal places. Find E from first and third readings and also from second and fourth readings.

Ans. 1,751,000 psi; 1,672,000 psi.

9-10. A 120-lb railroad rail 33 ft long is fastened securely at the ends in a track when the temperature is 100°F. The modulus of elasticity is 30,000,000 psi and the coefficient of expansion for steel is 0.0000067. If the ends do not "give" when the temperature drops to 0°, what is the total force on the ends required to hold the rail?

HINT: The weight of rails is given in pounds per yard, instead of pounds per foot as in the case of all other steel sections. *Ans.* 236,000 lb.

Contrary to some prevalent ideas, the modulus of elasticity is not always the same value for several specimens of a given material. Precision measurements show a slight variation on any one specimen. However, in most engineering design the modulus of elasticity is assumed constant up to the proportional limit. The average values given in Table 2 will be used in all problems unless otherwise stated.

TABLE 2. MODULUS OF ELASTICITY

(*This table should be memorized.*)

Material	*E, psi*
Steel	30,000,000
Wrought iron	27,000,000
Cast iron	15,000,000
Timber (parallel to the grain)	1,200,000
Portland cement concrete	2,000,000
Aluminum	10,600,000

10. Work and Resilience. When force acts on a body and the point of the body at which the force is applied moves in the direction of the force, the force is said to do *work*. The distance which the point of application of the force moves is called the *displacement*. The

component of the *displacement* in the direction of the force is the *effective displacement.* If the magnitude of the force is represented by P and the effective displacement is represented by x, work $= P \times x$. If the force is in pounds and the displacement is in feet, the work is expressed in *foot-pounds.* If the force is not constant, the work is the product of the *average force* multiplied by the displacement. When an elastic body is deformed, the force varies directly as the displacement (provided the elastic limit is not exceeded) and the average force is half the sum of the initial and final forces.

Example 1

A helical spring is stretched 1 in. by a load of 12 lb. What force will stretch the spring 3 in.? What is the average force for the elongation of 3 in.? What is the work done in stretching the spring 3 in.?

The force required to stretch the spring 3 in. is $P = 12 \times 3 = 36$ lb. The average force for the first interval of 3 in. is $(0 + 36)/2 = 18$ lb.

The average force is the force at the middle of the interval, which is an elongation of 1.5 in. Average force equal $1.5 \times 12 = 18$ lb.

$$\text{Work of displacement} = 18 \times 3 = 54 \text{ in.-lb} = 4.5 \text{ ft-lb.}$$

Example 2

After the spring of Example 1 has been stretched 3 in., an additional force is applied which produces an additional elongation of 4 in. What is the additional force? What is the average force while the spring is stretched the last 4 in.? What is the work done in stretching the spring the last 4 in.?

Ans. Additional force = 48 lb; average force = 60 lb; work = 20 ft-lb.

Problems

10-1. Find the work done in stretching the foregoing spring from 0 to 4 in. and then from 4 to 7 in.

10-2. A load of 36,000 lb is applied gradually to a steel rod which has no initial load. The elongation is 0.03 in. Find the work in foot pounds. *Ans.* 45 ft-lb.

10-3. An additional load of 24,000 lb is applied to the rod of Prob. 10-2. If the stress does not exceed the elastic limit, find the additional work in foot-pounds. *Ans.* 80 ft lb.

10-4. Find the total work on the rod of Prob. 10-2 when a load of 60,000 lb is gradually applied.

10-5. The rod of Prob. 10-2 is 2 in. square and the modulus of elasticity is 30,000,000 psi. What is the length? What is the elastic energy per cubic inch when the total load is 60,000 lb? *Ans.* 3.75 in.-lb per cu in.

11. Modulus of Resilience. The work expended by uniaxial forces in deforming *unit volume* of any solid to the *elastic limit* is called the *modulus of resilience* of the material. It is the *elastic potential energy* of unit volume when stressed to the elastic limit. The modulus of

resilience is a measure of the amount of elastic energy which may be stored in unit volume of a given material and recovered as mechanical work without loss.

If a unit cube of a solid is subjected to unit stress s, the deformation is s/E and the total work is

$$u = \frac{s}{2} \times \frac{s}{E} = \frac{s^2}{2E} \qquad \text{Formula IV}$$

in which u is the work done in deforming the volume or the stored potential energy.

Formula IV gives the elastic energy at any stress *below* the elastic limit. For the particular value of s at the elastic limit, the expression represents the *modulus* of *resilience.*

When s and E are given in pounds per square inch, Formula IV gives energy in inch-pounds per cubic inch.

If all parts of a solid body are subjected to the same unit stress s, the total elastic energy is obtained by multiplying the total volume of the body by the energy per unit volume.

The increase in energy per unit volume when the unit stress changes from s_1 to s_2 is expressed by the equation

$$u_2 - u_1 = \frac{s_2^2 - s_1^2}{2E} \qquad (11.1)$$

in which $\frac{s_2^2}{2E}$ is the total energy when the unit stress is s_2 and $\frac{s_1^2}{2E}$ is the total energy when the unit stress is s_1. The difference as given by Eq. (11.1) represents the work done in changing the stress of unit volume from s_1 to s_2.

Equation (11.1) may be checked by using the total forces and total deformations. Let P_1 be the total load corresponding to s_1, and P_2 the load corresponding to s_2. Then the average force times the displacement gives the total work done on the gage length L.

$$\text{Work} = \frac{(P_1 + P_2)}{2} e \qquad (11.2)$$

When this is divided by the cross-section area and the length of the specimen, the result should check Eq. (11.1) above.

Example

A 5- by 4- by 12-in. wood specimen when tested in compression parallel to the length shortens 0.006 in. in an 8-in. gage length when the total load changes from 4,000 to 22,000 lb. Find the modulus of elasticity and the unit energy stored up in the gage length by this change in load.

$$s_1 = 200 \text{ psi} \qquad s_2 = 1{,}100 \text{ psi} \qquad \epsilon = \frac{0.006}{8} = 0.00075$$

$$E = \frac{1{,}100 - 200}{0.00075} = 1{,}200{,}000 \text{ psi}$$

$$\text{Total energy} = U = \text{average force} \times \text{displacement} = \frac{4{,}000 + 22{,}000}{2} \times 0.006$$

$$= 78 \text{ in.-lb}$$

$$\text{Unit energy} = u = \frac{78}{\text{volume}} = \frac{78}{5 \times 4 \times 20} = 0.4875 \text{ in.-lb per cu in.}$$

Check by Eq. (11.1):

$$u = \frac{1100^2 - 200^2}{2 \times 1{,}200{,}000} = 0.4875 \text{ in.-lb per cu in.}$$

Problems

11-1. Find the modulus of resilience for steel having a modulus of elasticity of 30,000,000 psi and an elastic limit of 60,000 psi. *Ans.* 60 in.-lb per cu in.

11-2. What is the modulus of resilience of timber for which the modulus of elasticity is 1,200,000 psi and the elastic limit is 3,600 psi? How high would the elastic energy lift its own weight if the density of this timber is 36 lb per cu ft?

11-3. A piece of timber, 2 in. square and 5 ft long, is shortened 0.096 in. in a length of 4 ft when the load is changed from 4,800 to 12,800 lb. Find the total work in the gage length. *Ans.* U = 844.8 in.-lb.

11-4. From the answer of Prob. 11-3 calculate the work per cubic inch. Check by Eq. (11.1).

11-5. A ¾-in. diameter specimen when tested in tension elongated 0.0036 in. in an 8-in. gage length when the total load increased from 442 to 6,409 lb. From the total loads and total strain find the total work done on the gage length, and hence the unit work. Check the energy by Eq. (11.1).
Ans. 3.4875 in.-lb per cu in.

11-6. A 4- by 6- by 10-in. specimen, when tested in compression parallel to its length, shortens 0.004 in. in an 8-in. gage length when the total load increases from 4,800 to 31,200 lb. Find the modulus of elasticity. Find the total work done on the gage length by this increase in load. Divide by the volume and check by the formula.

12. Poisson's Ratio. A body subjected to tensile stress is elongated and the magnitude of the elongation, provided the elastic limit is not exceeded, is proportional to the unit stress. At the same time the dimensions at right angles to the direction of the tensile stress become smaller. A body under compressive load is shortened in the direction of the load, while its transverse dimensions are increased. The ratio of the unit deformation at right angles to the direction of the load to the unit deformation in the direction of the load is called *Poisson's ratio*. Since this ratio is approximately 0.25 for some common materials and since very exact measurements are required for its determination, Poisson assumed that the value is always ¼. In reality, Poisson's

ratio varies from below 0.15 for concrete to over 0.40 for hard rubber. For steel and steel alloys with high elastic limit, Poisson's ratio is about 0.26. If a bar of steel is elongated 0.001 of its original length by a tensile force, its transverse dimensions are reduced by about 0.00026 of their original value.

Poisson's ratio is represented in this book by the Greek letter μ (mu). When not otherwise specified the value of $\frac{1}{4}$ will be used here.

Problems

12-1. A steel rod 2 in. in diameter is stretched 0.0160 in. in a gage length of 20 in. Its diameter is reduced 0.00043 in. Find Poisson's ratio. *Ans.* 0.27.

12-2. If Poisson's ratio is 0.26 and the modulus of elasticity of structural steel is 29,000,000 psi, how much is the width of a 6- by 1-in. steel bar decreased by a pull of 121,800 lb? *Ans.* 0.00109 in.

12-3. Poisson's ratio for copper is about $\frac{1}{3}$ and the modulus of elasticity is 16,000,000. How much is the width of a plate, originally 8 in. wide and 0.162 in. thick, decreased by a pull of 12,960 lb?

12-4. A rod of 0.49% carbon steel tested by J. McLean Jasper[1] was 0.749 in. in diameter. When the load changed from 1,135 to 6,784 lb, the unit longitudinal deformation changed 0.000421 and the unit transverse dimension changed 0.000099. Find E and Poisson's ratio from this test.
Ans. $\mu = 0.235$.

12-5. A cylindrical core of concrete, tested by Dean A. N. Johnson,[2] was about 9 in. long, 4.5 in. in diameter, and 12 months old. When the load changed from 100 psi to 900 psi, the unit longitudinal deformation changed from -0.000026 to -0.000286 and the unit transverse deformation changed from 0.000005 to 0.000040. Find E and Poisson's ratio.
Ans. $E = 3{,}077{,}000$ psi; $\mu = 0.134$

12-6. A steel plate is subjected to a tensile stress of 12,000 psi parallel to the X axis and a tensile stress of 8,400 psi parallel to the Y axis. If the constant E is 30,000,000 psi and Poisson's ratio is $\frac{1}{4}$, what is the unit deformation parallel to each axis?

	Unit deformation
Ans. X	0.00033
Y	0.00018
Z	−0.00017

12-7. Solve Prob. 12-6 if the unit stress parallel to the Y axis is compressive.

Ans. X	0.00047
Y	−0.00038
Z	−0.00003

12-8. In Fig. 9 is shown a homogeneous and isotropic rectangular block on which are applied six forces uniformly over the areas. If the material is aluminum

[1] *Trans. ASTM*, Vol. 24, Part II, p. 1015, 1924.
[2] *Ibid.*, p. 1030.

with a modulus of elasticity of 10,000,000 psi and a Poisson's ratio of 1/3, what will be the total changes in the dimensions of the block caused by the application of the forces? *Ans.* +0.0034 in.; −0.0023 in.; +0.0021 in.

12-9. Solve Prob. 12-8 with the 60,000-lb forces as compression.

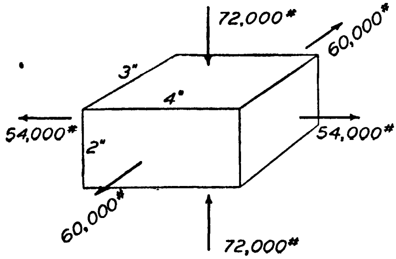


FIG. 9.

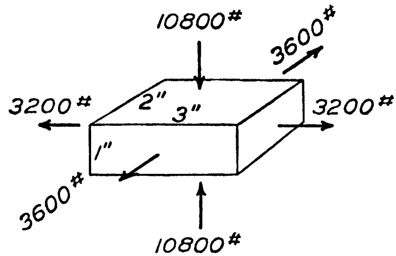


FIG. 10.

12-10. The parallelepiped shown in Fig. 10 is subjected to forces which are uniform over the areas. The material has a modulus of elasticity of 1,000,000 psi and a Poisson's ratio of 1/4. Find the change in the three principal dimensions. *Ans.* +0.00495 in.; −0.0025 in.; +0.0025 in.

13. Biaxial Stresses. Problems 12-6 and 12-7 are examples of *biaxial* stresses. The loads are applied parallel to the X and Y axes, and the deformation along any axis depends upon the stress along each axis. The calculation of the deformations when the stresses are given is a simple matter. On the other hand, it is sometimes desirable to compute the stresses from the unit deformations.

When the loads are applied parallel to the X and Y axes,

$$\epsilon_x = \frac{1}{E}(s_x - \mu s_y) \tag{13.1}$$

$$\epsilon_y = \frac{1}{E}(s_y - \mu s_x) \tag{13.2}$$

in which ϵ_x and ϵ_y are the unit deformations along the X and Y axes, respectively.

When the unit deformations are known and the unit stresses are to be calculated, elimination of s_y gives

$$s_x(1 - \mu^2) = E(\epsilon_x + \mu\epsilon_y) \tag{13.3}$$

$$s_x = E\frac{\epsilon_x + \mu\epsilon_y}{1 - \mu^2} \tag{13.4}$$

By symmetry,

$$s_y = E\frac{\epsilon_y + \mu\epsilon_x}{1 - \mu^2} \tag{13.5}$$

When $\epsilon_x = \epsilon_y$,

$$s_x = s_y = E\frac{\epsilon_x}{1-\mu} \tag{13.6}$$

Problems

13-1. A steel plate is subjected to tension along the X and the Y axes. The unit deformation along each of these is 0.00037. If E = 30,000,000 psi and Poisson's ratio is 0.26, what is the unit stress along the X and Y axes, and the unit deformation along the Z axis?
Ans. $s_x = s_y$ = 15,000 psi; ϵ_z = −0.00026.

13-2. The unit elongation in a steel plate along the X axis is 0.00054 and along the Y axis is 0.00036, while Poisson's ratio is 0.25. Solve.
Ans. s_x − 20,160 psi; s_y = 15,840 psi; ϵ_z = −0.00030.

13-3. Solve Prob. 13-2 if the deformation along the Y axis is negative.
Ans. s_x = 14,400 psi; s_y = −7,200 psi; ϵ_z = −0.00006.

13-4. The unit deformation along the X axis is +0.00040 and along the Y axis is +0.00032. Find the unit stresses for steel. *Ans.* 15,360 psi; 13,440 psi.

13-5. The unit deformation along the X axis is +0.00036 and along the Y axis is +0.00020. Find the stresses for steel.

13-6. The unit deformation along the X axis is +0.00050 and along the Y axis is −0.00024. Find the stresses for steel. *Ans.* 14,080 psi; −3,680 psi.

13-7. The unit deformation along the X axis is −0.00040 and along the Y axis is +0.00028. Find the stresses for steel.

The energy stored up in a cubic inch of material when subjected to a single axial tensile stress s_x is the average stress times the unit deformation

$$u = \frac{0 + s_x}{2}\epsilon_x = \frac{s_x^2}{2E} \tag{13.7}$$

When a cubic inch of material is subjected to two tensile stresses on orthogonal axes,

$$u = \frac{0 + s_x}{2}\epsilon_x + \frac{0 + s_y}{2}\epsilon_y \tag{13.8}$$

$$u = \frac{1}{2E}(s_x^2 + s_y^2 - 2\mu s_x s_y) \tag{13.9}$$

Problems

13-8. The elastic limit of a steel is 36,000 psi. Find the unit energy stored up when s_x = 36,000 psi and s_y = 0. What is this called?
Ans. 21.6 in.-lb per cu in.

13-9. Solve Prob. 13-8 if s_x and s_y are each 36,000 psi.
Ans. 32.4 in.-lb per cu in.

13-10. Solve Prob. 13-9 if s_x is 36,000 psi tension and s_y = 36,000 psi compressive.
Ans. 54.0 in.-lb per cu in.

14. Triaxial Stresses. The theory of volume elasticity in Art. 15 involves *triaxial* stresses. Triaxial stresses occur in the walls of a tank subjected to internal liquid pressure. An *axial* tensile stress parallel to the length resists the opposite pressures on the heads, which tend to rupture the tank around any circumference. A *circumferential* tensile stress resists the opposite pressures on any two halves of the side walls, which tend to split the tank longitudinally. Near the inner surface a *radial* compressive stress resists the normal pressure of the liquid on the inner surface and the radial components of the circumferential stress in the material outside the element under consideration. The approximate calculations of *thin-walled* tanks and boiler tubes are simple (see Arts. 42 and 43) but the calculations of *relatively thick-walled* tanks involve the principles of triaxial loading. Similar problems occur in the calculation of the stresses in boiler tubes and in cylinders of internal-combustion engines, in which large relative deformations are caused by the temperature difference between the inner and outer surfaces.

In the theory of volume elasticity, the stresses are given and the deformations are easily calculated. In order to find the stresses from the unit deformation, the equations must be transformed to express unit stress explicitly in terms of the unit deformations.

$$\epsilon_x = \frac{1}{E}(s_x - \mu s_y - \mu s_z) \tag{14.1}$$

$$\epsilon_y = \frac{1}{E}(-\mu s_x + s_y - \mu s_z) \tag{14.2}$$

$$\epsilon_z = \frac{1}{E}(-\mu s_x - \mu s_y + s_z) \tag{14.3}$$

From Eqs. (14.1) and (14.2), eliminating s_z,

$$\epsilon_x - \epsilon_y = \frac{1}{E}[(1 + \mu)s_x - (1 + \mu)s_y] \tag{14.4}$$

From Eqs. (14.2) and (14.3), eliminating s_z,

$$\epsilon_y + \mu\epsilon_z = \frac{1}{E}[-(\mu + \mu^2)s_x + (1 - \mu^2)s_y] \tag{14.5}$$

Multiplying Eq. (14.4) by μ and adding to Eq. (14.5) eliminates s_x.

$$\mu\epsilon_x + (1 - \mu)\epsilon_y + \mu\epsilon_z = \frac{1}{E}(1 - \mu - 2\mu^2)s_y \tag{14.6}$$

$$s_y = E\frac{\mu(\epsilon_x + \epsilon_z) + (1 - \mu)\epsilon_y}{(1 + \mu)(1 - 2\mu)} \tag{14.7}$$

By symmetry,

$$s_x = E\frac{\mu(\epsilon_y + \epsilon_z) + (1 - \mu)\epsilon_x}{(1 + \mu)(1 - 2\mu)}$$
$$s_z = E\frac{\mu(\epsilon_x + \epsilon_y) + (1 - \mu)\epsilon_z}{(1 + \mu)(1 - 2\mu)} \tag{14.8}$$

Problems

14-1. Material for which E = 30,000,000 psi and Poisson's ratio = 0.24 is stressed in three directions. The deformation in 8 in. is 0.008192 in. along the X axis and 0.002240 in. along the Y axis. The deformation along the Z axis is -0.001672 in. in a gage length of 2 in. Find the unit stress in each direction. *Ans.* $s_z = -15{,}000$ psi.

14-2. If s_z is known to be zero, find the values of s_x and s_y in terms of ϵ_x and ϵ_y and compare with Eq. (13.4).

14-3. If s_z is known to be zero and the unit deformations along the X and Y axes are the same, find s_x and compare with Eq. (13.6).

14-4. In a steel plate the three principal deformations are: along X -0.00048, along Y $+0.00036$, and along Z $+0.00020$. Find the three corresponding stresses. *Ans.* $-10{,}560$ psi; $+9{,}600$ psi; $+5{,}760$ psi.

14-5. If the three unit deformations in a steel plate are $X = +0.00040$, $Y = -0.00024$, and $Z = -0.00016$, find the stresses.

The energy stored up in a cubic inch of material when subjected to three tensile stresses on orthogonal axes is

$$u = \frac{s_x}{2}\epsilon_x + \frac{s_y}{2}\epsilon_y + \frac{s_z}{2}\epsilon_z \tag{14.9}$$

$$u = \frac{1}{2E}[s_x^2 + s_y^2 + s_z^2 - 2\mu(s_xs_y + s_ys_z + s_zs_x)] \tag{14.10}$$

Problems

14-6. If a steel is subjected to $s_x = s_y = s_z = 36{,}000$ psi, find the unit energy absorbed. *Ans.* 32.4 in.-lb per cu in.

14-7. The unit stresses which are applied to an element are $s_x = s_y = 36{,}000$ psi, and $s_z = -36{,}000$ psi. Find the energy in a cubic inch.

14-8. The unit stresses which are applied to an aluminum element are $s_x = 15{,}000$ psi, $s_y = s_z = -15{,}000$ psi. Poisson's ratio is $\frac{1}{3}$. Find the unit energy absorbed.

15. Volume Change and Modulus of Elasticity. When a solid is subjected to a load in one direction, there is a slight change in volume. The relative change in area of cross section at right angles to the load is smaller than the unit deformation in the direction of the load. Consequently, when the load is compressive, the volume is reduced; and when the load is tensile, the volume is increased.

Problems

15-1. A steel bar, 2 in. square and 10 in. long, is subjected to a compressive load of 96,000 lb in the direction of its length. If $E = 30{,}000{,}000$ psi and Poisson's ratio is 0.27, what are the length, area of cross section, and volume when the load is on? *Ans.* 9.992 in.; 4.001728 sq in.; 39.98527 cu in.

15-2. Find the work done by the load of Prob. 15-1. Find the work per unit volume two ways. *Ans.* 32 ft-lb; 0.8 ft-lb per cu in.

15-3. A round rod, 2 in. in diameter and 20 in. long, has its diameter reduced 0.0005 in. and its volume increased 0.0228 cu in. by a load of 78,540 lb. Find E and Poisson's ratio.

If a unit cube is elongated an amount ϵ by an external pull, its length becomes $1 + \epsilon$ and its transverse dimensions become $1 - \mu\epsilon$, in which μ is Poisson's ratio.

$$\text{Area of cross section} = (1 - \mu\epsilon)^2 = 1 - 2\mu\epsilon + (\mu\epsilon)^2 \qquad (15.1)$$

Since $\mu\epsilon$ is small, never being greater than 0.001, its square, which is relatively much smaller, may be neglected; hence the approximate cross section is

$$A = 1 - 2\mu\epsilon \qquad (15.2)$$

Multiplication of area by length gives

$$\text{Volume} = (1 - 2\mu\epsilon)(1 + \epsilon) = 1 + (1 - 2\mu)\epsilon - 2\mu\epsilon^2 \qquad (15.3)$$

of which the last term, $2\mu\epsilon^2$, may be neglected.

$$\text{Approximate volume} = 1 + (1 - 2\mu)\epsilon \qquad (15.4)$$

After the original volume of one cubic unit has been subtracted from the approximate volume under tension, the remainder gives

$$\text{Increment of volume} = (1 - 2\mu)\epsilon \qquad (15.5)$$

These formulas apply only to temporary deformations below the elastic limit. The permanent deformations which occur when the elastic limit is exceeded produce practically no change of volume.

Problems

15-4. If the external force is compressive, show that

$$\text{Approximate volume} = 1 - (1 - 2\mu)\epsilon$$
$$\text{Increment of volume} = -(1 - 2\mu)\epsilon$$

15-5. A block of hard steel, originally 2 in. square and 10 in. long, is subjected to a load of 144,000 lb parallel to its length. If $E = 30{,}000{,}000$ and Poisson's ratio is 0.27, what is the increment of cross section and the total increment of volume? *Ans.* 0.002592 sq in.; −0.02208 cu in.

15-6. Find the area and volume of the block of Prob. 15-5 to eight decimal places by direct multiplication without the use of the foregoing equations. Find the increments of area and volume and compare with the answers of Prob. 15-5.

A solid submerged in a liquid is under pressure from all directions. The quotient obtained when the unit pressure is divided by the relative reduction of volume is called the *modulus of volume elasticity*. If, for instance, 1 cubic inch of a solid is reduced to 0.9995 cubic inch by a pressure of 10,000 pounds per square inch in all directions, the modulus of volume elasticity is

$$K = \frac{10{,}000}{0.0005} = 20{,}000{,}000 \text{ psi}$$

Problem

15-7. A block of steel has its volume changed from 40.320 to 40.200 cu in. by a liquid pressure of 60,000 psi. Find the modulus of volume elasticity.

Ans. $K = 20{,}160{,}000$ psi.

The modulus of volume elasticity may be computed from the modulus of linear elasticity (Young's modulus) and Poisson's ratio. If a cube of unit dimensions is subjected to unit pressure s in the direction of any axis, it is shortened s/E in the direction of the pressure and elongated $\mu s/E$ along each of the two axes at right angles to the direction of the pressure. When there is a compressive stress s in every direction, the compression along any axis is made up of the direct compression s/E, which is due to the pressure in that direction, and two elongations, each of magnitude $\mu s/E$, which are due to pressures along the two axes at right angles with the first. (It is assumed that the body is *isotropic*, having the same properties in every direction, so that E and μ are the same for all axes.)

In any direction,

$$\text{Total compression} = \frac{s}{E} - \frac{2\mu s}{E} = \frac{s}{E}(1 - 2\mu) \qquad (15.6)$$

The length of each edge of the cube becomes $1 - \frac{s}{E}(1 - 2\mu)$.

$$\text{Volume} = \left[1 - \frac{s}{E}(1 - 2\mu)\right]^3$$

$$= 1 - \frac{3s}{E}(1 - 2\mu) + \frac{3s^2}{E^2}(1 - 2\mu)^2 + \cdots \qquad (15.7)$$

Since s/E is very small, the terms containing the higher powers may

be dropped and Eq. (15.7) becomes

$$\text{Final volume} = 1 - \frac{3s}{E}(1 - 2\mu) \tag{15.8}$$

Since the original volume was unity, the decrease in volume is

$$\frac{3s}{E}(1 - 2\mu)$$

which is also the unit change of volume or the unit volume deformation. The modulus of volume elasticity is obtained by dividing the unit stress s by the unit volume deformation.

$$K = \frac{s}{(3s/E)(1 - 2\mu)} = \frac{E}{3(1 - 2\mu)} \tag{15.9}$$

Problems

15-8. If Poisson's ratio is $\frac{1}{4}$, show that the modulus of volume elasticity is two-thirds the modulus of linear elasticity.

15-9. What would be the modulus of volume elasticity if Poisson's ratio were $\frac{1}{2}$?

15-10. If E = 15,500,000 psi and K = 10,200,000 psi, what is Poisson's ratio?
Ans. 0.255.

16. Miscellaneous Problems

16-1. A longleaf yellow-pine post, tested at Watertown Arsenal ("Tests of Metals," 1897, p. 415), was 9.79 by 9.81 in. The gage length was 50 in. When the total load changed from 19,210 to 211,290 lb, the compression in the gage length increased from 0.0035 to 0.0460 in. Calculate the area to two decimal places. Find E and the total work in the gage length. Divide the total work by the volume to get the work per cubic inch. Check.
Ans. E = 2,353,000 psi; u = 1.019 in.-lb per cu in.

16-2. A second post of longleaf yellow pine ("Tests of Metals," 1897, p. 417) was 9.76 by 9.79 in. When the load changed from the initial value of 1,911 to 191,100 lb, the measured compression in the gage length of 50 in. changed from 0 to 0.0568 in. Find E and the work per unit volume. Check.
Ans. E = 1,743,000 psi.

16-3. A stick of Douglas fir, tested in tension ("Tests of Metals," 1896, p. 405), was 24 ft $2\frac{5}{8}$ in. long, 8.12 in. wide, and 3.02 in. thick. The stick weighed 163 lb. When the load changed from 2,450 to 51,450 lb, a gage length of 200 in. elongated 0.1493 in. Find the area to the first decimal place and the weight per cubic foot. Find E and the total work in the gage length. Find the work per cubic inch and check.
Ans. E = 2,679,000 psi; u per cu in. = 0.8212 in.-lb.

16-4. A compression piece 60 in. long cut from the tension piece of Prob. 16-3 was shortened 0.0294 in. in a gage length of 50 in. when the load changed from 2,450 to 49,000 lb, and was shortened 0.0455 in. when the load changed from 2,450 to 73,500 lb. Find E and the work per unit volume for the two large intervals. Check.
Ans. E = 3,231,000 and 3,187,000 psi; u = 0.6174 and ?

16-5. In Prob. 16-4, when the load changed from 2,450 to 49,000 lb., a transverse gage length of 7 in. was elongated 0.0022 in., and when the load changed from 2,450 to 73,500 lb, the elongation was 0.0034 in. Find Poisson's ratio.

Ans. $\mu = 0.534$.

(Answer seems too large, but wood is not isotropic.)

16-6. A stick of Douglas fir ("Tests of Materials," 1896, p. 407), tested in tension, was stretched 0.2614 in. in a gage length of 200 in. when the unit stress changed from 100 to 2,400 psi while a transverse gage length of 12 in. shortened 0.0063 in. Find E and Poisson's ratio.

Ans. $E = 1{,}760{,}000$ psi; $\mu = 0.40$.

16-7. A compression piece, cut from the stick of Prob. 16-6, shortened 0.0471 in. in a gage length of 50 in. when the load changed from 100 to 2,000 psi. Find E. A transverse gage length of 12 in. elongated 0.0064 in. for the same change in load. Find Poisson's ratio.

Ans. $E = 2{,}170{,}000$ psi; $\mu = 0.566$.

16-8. A white-oak post ("Tests of Metals," 1896, p. 425) was 106.1 in. long, had a cross section of 9.95 by 11.98 in., and weighed 415.5 lb. A 50-in. gage length was shortened 0.0260 in. when the load changed from 11,920 to 119,200 lb. Find E. *Ans.* 1,731,000 psi.

16-9. A short block 11.97 by 8.10 in. cut from the post of Prob. 16-8 was loaded transversely, perpendicular to the growth rings. When the load changed from 1,939 to 38,784 lb, the compression in a 6-in. gage length was 0.0118 in. Find the modulus of elasticity of this oak across the grain.

Ans. $E = 193{,}000$ psi.

16-10. A paving brick, tested in compression lengthwise ("Tests of Metals," 1896, p. 359), was 8.14 by 2.46 by 4.18 in. When the load changed from 1,028 to 102,800 lb., the compression in a gage length of 5 in. was 0.0071 in. and the elongation in a transverse gage length of 3.5 in. was 0.0008 in. Find E and Poisson's ratio. *Ans.* $E = 6{,}970{,}000$ psi; $\mu = 0.161$.

16-11. An oak block, 6 in. square and 30 in. long, is bolted between two steel plates, each 6 in. wide, ½-in. thick, and 30 in. long. A force applied lengthwise of the combined block shortens it 0.008 in. in a length of 20 in. If E for the steel is 29,000,000 psi and E for the oak is 1,600,000 psi, what is the total force? *Ans.* 92,640 lb.

16-12. In Prob. 16-11, what is the unit stress in the steel when the unit stress in the oak is at its allowable value in compression?

16-13. A vertical pier, 26 in. square, is made of concrete in which four 4- by 3- by ½-in. structural-steel angles are embedded. The modulus of elasticity of the steel is fifteen times the modulus of the concrete. When the pier carries a load of 360,360 lb, how much of this load is carried by the concrete and how much is carried by the steel? What is the unit stress in each?

16-14. A vertical pier 6 in. square is made of concrete in which four 1-in. square steel rods are embedded as vertical reinforcing. Both the concrete and the rods are exactly 12 in. long. The combination is placed in a testing machine and a compressive load of 46,000 lb applied. Find the total shortening of the pier and the stresses in each material.

Ans. $S_c = 500$ psi; $S_s = ?$

16-15. A 12-in. (nominal size) standard steel pipe weighing 49.56 lb per ft is 20 in. long and stands vertically. The pipe is filled with concrete which under

these conditions may withstand a working stress of 1,000 psi. What total safe compressive load may be applied by a testing machine? Which material governs? What is the total shortening of the combination?

Ans. 331,800 lb.

16-16. A weightless wire is stretched between two supports and carries a horizontal cable weighing 16 lb per ft as shown in Fig. 11. The span is 200 ft and the sag of the wire at the middle is 12.31 ft. If the wire is nickel steel, find the diameter. *Ans.* 0.576 in.

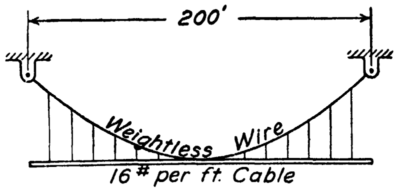


FIG. 11.

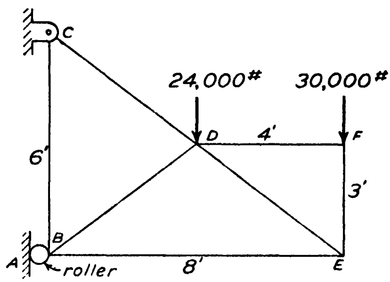


FIG. 12.

16-17. For the pin-connected truss shown in Fig. 12, find the diameter of a round steel bar to be used for each tension member. If *BD* is a square oak member, find its size. *Ans.* CD = 2.11 in. diameter.

16-18. Two steel I-bars are attached to pins at the ceiling and joined by a pin at *C* where a 42,000-lb load is hung, as shown in Fig. 13. Find the size of commercial I bars to carry the load safely. *Ans.* 3 by ⅝-in. will do for *BC*.

16-19. A steel rod 0.798 in. in diameter elongates 0.01040 in. in a gage length of 20 in. when its temperature is raised from 60 to 140°F. Tested in tension, the rod is stretched 0.0064 in. in a gage length of 8 in. when the pull

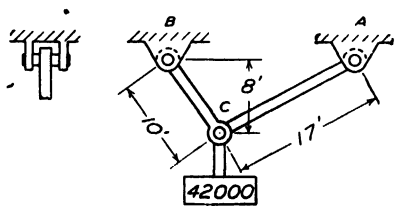


FIG. 13.

changed from 200 to 12,000 lb. Find *E* and the coefficient of linear expansion. The ends of the rod are fastened to a rigid frame and the temperature is lowered from 120 to 60°F. What is the increase in the total tension and in the unit tensile stress if the resistance of the frame entirely prevents the rod from contracting? Solve also if a 20-in. gage length shortens 0.0028 in. with the fall of temperature.

Ans. E = 29,500,000 psi; temperature coefficient = 0.0000065 per degree Fahrenheit; 11,505 psi; 5,752 lb; 7,375 psi; 3,687 lb.

16-20. In Fig. 14, A, B, C, D, and E are points on a vertical wall. A and B are on the same horizontal level and C is vertically below E. K is 4 ft directly in front of D. AK and BK are weightless aluminum rods hinged at A, B, and K. The strut CK is hinged at both ends also. The 31,500-lb load at K is vertical. Find the required areas of the rods AK and BK.

Ans. AK is 1.4 sq in.

16-21. A steel bar in the form of a frustum of a pyramid is 1 in. square at one end, 2 in. square at the other end, and 10 in. long. A load of 30,000 lb is applied in compression. If E is 30,000,000 psi and if it is assumed that the stress in any transverse section is uniform throughout the section, calculate the decrease in length by means of integral calculus. Compare with a uniform bar 1.5 in. square. *Ans.* Total compression is 0.005 in.

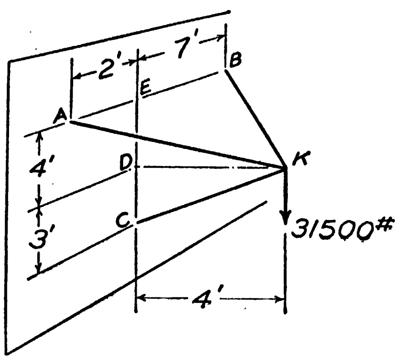


FIG. 14.

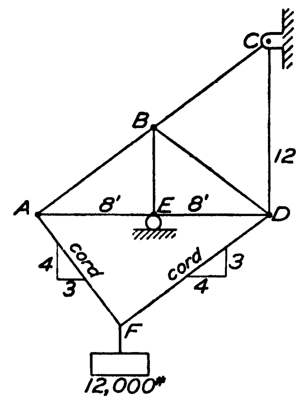


FIG. 15.

16-22. By integration, find the total internal work of the bar of Prob. 16-21.

Ans. 75 in.-lb.

16-23. The total external work of the bar of Prob. 16–21 is the product of the total compression multiplied by the average load. Solve for the external work by means of the answer of Prob. 16–21 and compare with the internal work.

16-24. A bar 1 in. thick and 20 in. long is 1 in. wide at one end and increases uniformly to a width of 3 in. at the other end. If a load of 30,000 lb in compression is applied to the bar, find the unit stress at a distance x from the small end. If E is 30,000,000 psi, find the unit deformation and find the total deformation in the entire length. *Ans.* 0.01986 in.

16-25. By integration, solve Prob. 16-24 for the total internal work and then check Prob. 16-24 by means of the external work.

16-26. The pin-connected truss $ABCDE$ shown in Fig. 15 is hinged to the wall at C and supported on a roller at E. Two cords, AF and DF, support the 12,000 lb load. If the members of the truss are made of aluminum, find the area of each tension member of the truss. *Ans.* AB requires 0.853 sq in.

CHAPTER 2

SHEAR

17. Shear and Shearing Stress. When a body is subjected to a pair of forces which are in the *same line* and directed *away* from each other, *tensile* stress is produced. When the pair of forces is in the *same line* and directed *toward* each other, *compressive* stress is produced. If the forces are in *parallel lines* or *planes*, *shearing* and bending stresses are produced in the portion of the body between them. In Fig. 16, the block A is securely held by the clamp B and a horizontal force P is applied by a block C. The force P is parallel to the upper surface of B. The clamp B exerts a horizontal force on the block A. This force is equal and opposite to the force P. The portion of the block A between the upper surface of the clamp and the lower surface EFG of the block C is subjected to a pair of equal, opposite, parallel forces. The material in this portion of the block is subjected to shearing and bending stresses. The shearing stresses depend upon the magnitude of the forces and the area of the section of A. The bending stresses depend upon these and also upon the distance of the forces apart. If the body C is brought very close to B, so that the distance between the two forces P and P' becomes negligible, the unit bending stress becomes small, while the unit shearing stress is unchanged. The average unit shearing stress is calculated by dividing the force P by the area of the cross section EFG or the area of any section parallel to it.

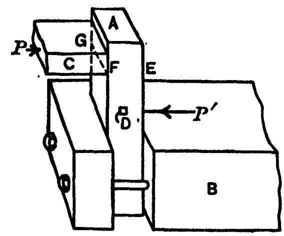


FIG. 16. Shear and bending.

In tension and compression the unit stress is calculated by dividing the total force by the area of the cross section perpendicular to it. In shear, on the other hand, the unit stress is calculated by dividing the total force by the area of the cross section parallel to it.

In Fig. 16, as in all cases of application of force, the line P represents the resultant of a set of forces distributed over an area. The resultant P' must fall some distance below the upper surface of B and the resultant P must lie above the lower surface of C. It is, therefore, not prac-

ticable to secure shearing stress entirely free from bending or compressive stress. It will be shown later that the distribution of shearing stress, when combined with bending, is not uniform over the section. At present, however, no account will be taken of this variation, and the average shearing stress will be calculated by dividing the total force by the area in shear.

TABLE 3. ALLOWABLE UNIT SHEARING STRESS
(*This table should be memorized*)

Material	*Psi*
Steel rivets	15,000
Pins and turned bolts in reamed or drilled holes	15,000
Unfinished bolts	10,000
Webs of beams and girders	13,000
Weld metal, net section of fillet weld	13,600
Weld metal, net section of butt weld	13,000
Aluminum 17S-T and 24S-T	10,000
Timber, parallel to the grain	100

Problems

17-1. Two 3- by ½-in. plates are united by one ¾-in. power-driven rivet. What is the allowable load in shear? *Ans.* 6,630 lb.

17-2. One 3- by ½-in. plate is placed between two 3- by ⅜-in. plates and connected by one ⅞-in. unfinished bolt which passes through all three plates. What is the allowable load in shear? *Ans.* 12,020 lb.

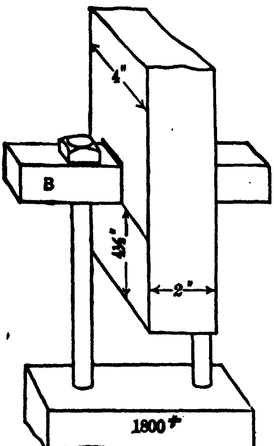


FIG. 17. Shear in timber.

17-3. A 5- by 1-in. I bar (see handbook) has one end between two plates. With 20,000 psi as the allowable tensile unit stress in the I bar, what is the minimum allowable diameter of the pin in double shear which connects it with the plates? *Ans.* 2.92 in.

17-4. A 2- by 4-in. yellow-pine block (Fig. 17) hung vertical and supported at the upper end, has a hole 1.2 in. square, which is perpendicular to the 4-in. faces. The lower edge of this hole is 4½ in. above the lower end of the block. If a load of 1,800 lb is hung on a square bar passing through this block, what is the unit shearing stress in the pine parallel to the grain? *Ans.* 100 psi.

17-5. What is the unit tensile stress at the minimum section of Fig. 17 when the load is 1,800 lb?

17-6. The head of a 1½-in. American Standard regular bolt is 1 in. high. A pull of 20,000 lb applied to the bolt and resisted by the head tends to shear the head from the body of the bolt. Find the unit shearing stress. Find the unit tensile stress in the gross section of the bolt. If the force is

applied at the opposite end of the bolt by means of a nut, find the unit tensile stress at the minimum section (see AISC handbook).

Ans. $s_s = 4{,}244$ psi; $s_t = 11{,}320$ psi; $s_t = 15{,}460$ psi.

17-7. The load of 20,000 lb is applied to the bolt of Prob. 17-6 by means of a nut. Find the unit shearing stress at the root of the threads.

Ans. $s_s = 4{,}320$ psi.

17-8. Solve Probs. 17-6 and 17-7 for a $1\frac{3}{4}$-in. bolt subjected to a 25,000 lb force. HINT: This bolt requires an American Standard heavy nut.

17-9. In Fig. 13 find the diameter of a fitted steel pin for the support at A. Find also the diameter for B. *Ans.* 1.05 in. diameter.

17-10. In Fig. 7 the fitted pin at J is in double shear. Find the required diameter for steel.

17-11. In Fig. 8 find the diameter of a fitted aluminum pin at joint G if the pin is in double shear. *Ans.* 1.54 in. diameter.

18. Shearing Deformation. In Fig. 16, a small portion of section D extends through the block A with its long dimension perpendicular to the plane which contains the resultants P and P'. The cross section D is represented on a large scale by the rectangle $HIJK$ of Fig. 18.

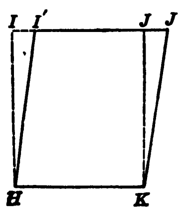


FIG. 18. Shearing deformations.

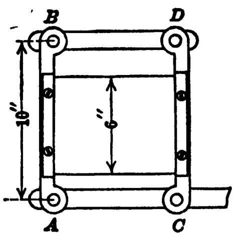


FIG. 19. Device for illustrating shear.

When the shearing forces are applied as shown in Fig. 16, this rectangle is distorted to the form $HI'J'K$. If the lower line, HK, is regarded as fixed, the total displacement of any point in the upper line is II' or JJ'. The unit shearing deformation, which may be represented by γ, (gamma) is the ratio of this horizontal displacement II' to the vertical distance HI. In linear deformation, the unit deformation is obtained by dividing the total deformation by a length in the *same direction as the deformation;* in shearing deformation, the displacement is divided by a distance at *right angles* to the *displacement.* The unit displacement is the tangent of the angle IHI'. The effect of the shearing forces is to lengthen the diagonal HJ and shorten the diagonal IK. Note that $\tan IHI' = II'/HI$, but since the angle IHI' is always very small,

$$\gamma = \frac{\text{total deformation}}{\text{length}}$$

Problems

18-1. Two equal bars, AB and CD (Fig. 19) are hinged to a second pair of equal bars, AC and BD, to form a parallelogram. A sheet of rubber, 6 in. wide, has one edge securely clamped to AB and the other edge to CD. The length of AB, center to center of hinges, is 10 in. What is the unit shearing displacement when B is displaced 0.2 in. to the right of the vertical?
Ans. Unit shear $\gamma = 0.02$.

18-2. A shaft 4 in. in diameter is twisted 3° in a length of 10 ft. What is the total displacement of a point on the surface at one end if the other end is regarded as fixed? What is the unit displacement?
Ans. 0.10472 in.; 0.000873 in.

19. Modulus of Elasticity in Shear. The modulus of elasticity in shear is obtained by dividing the unit shearing stress by the unit shearing deformation, just as the modulus of elasticity in tension or compression is computed by dividing the unit tensile or compressive stress by the corresponding unit deformation.

$$G = \frac{s_s}{\gamma} \tag{19.1}$$

The modulus of shearing elasticity is frequently called the *modulus of rigidity*.

Forces applied as in Fig. 16 do not give pure shear. Even in Fig. 17, in which the plates which apply the parallel forces are as close together as possible, shear is combined with bending. Pure shear, free from bending or compression, may be secured by torsion, as in Prob. 18-2.

When not otherwise specified, the following values of the modulus of rigidity will be used:

(These constants should be memorized.)

Steel = 12,000,000 psi
Aluminum = 4,000,000 psi

Problems

19-1. In Prob. 18-2, if G is 11,200,000 psi, what is the unit shearing stress at the surface of the shaft? *Ans.* $s_s = 9{,}778$ psi.

19-2. What is the maximum unit shearing deformation if the maximum allowable shearing stress is 10,000 psi, and the modulus of rigidity is 11,400,000 psi.
Ans. 0.000877.

19-3. A hollow shaft has an inside diameter of 4 in. and an outside diameter of 6 in. The shaft is twisted 0.02 rad in a length of 10 ft. What is the unit deformation at the inner surface? At the outer surface? What is the unit stress at each surface if the modulus of rigidity is 10,800,000 psi?
Ans. 3,600 psi at inner surface.

19-4. What would be the unit shearing deformation and unit stress in Prob. 19-3 at a point on the outer surface and 5 ft from the ends?

20. Shear Caused by Compression or Tension. Figure 20 shows a block subjected to a downward compressive force P in the direction of its length and an equal upward force at the bottom. This block may be supposed to be cut by a plane $BCDE$, normal to its length, and then glued together. If the portion above this section is regarded as a free body in equilibrium, and if the weight of the portion is neglected, the downward force P must be equal to the upward reaction of the glued

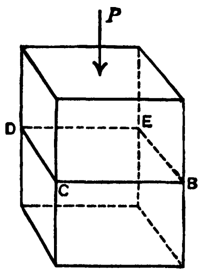


FIG. 20. Section normal to force.

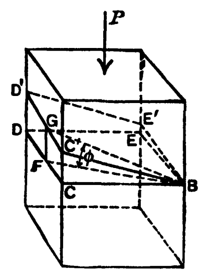


FIG. 21. Section inclined to force.

surface. If A is the area of the section, the unit compressive stress in the glue is given by

$$s_c = \frac{P}{A} \tag{20.1}$$

Since the external force P has no horizontal component, the shearing force in the glue is zero. If the body were actually made of two portions, the upper portion would not slide on the lower portion, no matter how smooth the surface of contact.

Figure 21 represents a body similar to Fig. 20, loaded and supported in the same way. This body is assumed to be cut by a plane $BC'D'E'$, which makes an angle ϕ with the normal section. The portion above the inclined section may be taken as the free body, and the external force P may be resolved perpendicular and parallel to this plane. The component of P normal to the plane is $P \cos \phi$. The unit compressive stress is this component divided by the area of the section. If A is the area of the normal section, the area of the inclined section is $A \sec \phi$. The unit compressive stress is given by

$$s_c = \frac{P \cos \phi}{A \sec \phi} = \frac{P}{A} \cos^2 \phi = \frac{P}{2A} (1 + \cos 2\phi) \tag{20.2}$$

The component of the force P in the direction of the line BG, which makes the maximum angle with the normal plane, is $P \sin \phi$. This component is resisted by the shearing stress in the section $BC'D'E'$.

The unit shearing stress is obtained by dividing the component parallel to the section by the area of the section.

$$s_s = \frac{P \sin \phi}{A \sec \phi} = \frac{P}{A} \sin \phi \cos \phi = \frac{P}{2A} \sin 2\phi \qquad (20.3)$$

If the body were in tension instead of compression, Eq. (20.3) would still give the unit shearing stress in the section, and Eq. (20.2) would give the unit tensile stress (instead of the unit compressive stress) normal to the section.

Problems

20-1. A 6- by 4-in. post is cut by a plane which makes an angle of 35° with the 6-in. faces and is normal to the 4-in. faces. What is the length of the intersection of this plane with the 4-in. faces? If a load of 10,800 lb is placed on this post, what is the component parallel to this plane? What is the component perpendicular to this plane? What is the unit shearing stress along this plane? What is the unit compressive stress perpendicular to the plane?
Make a sketch. Solve completely. Then check by Eqs. (20.2) and (20.3)
Ans. $s_c = 148$ psi.; $s_s = 211$ psi.

20-2. Solve Prob. 20-1 if the plane makes an angle of 55° with the 6-in. faces.

20-3. Show from Eqs. (20.2) and (20.3) that the shearing stress is zero and the compressive stress is a maximum when $\phi = 0$. Explain from your sketch.

20-4. A 6- by 6-in. post is subjected to a load of 10,800 lb in the direction of its length. Find the unit shearing stress and the unit compressive stress with respect to a plane which makes an angle of 25° with the normal section.

20-5. Show that the unit shearing stress produced by a single tensile or compressive load is a maximum at 45° with the direction of the load, and that this maximum shearing stress is one-half the unit tensile or compressive stress which produces it.

20-6. A timber block is 4 by 4 by 18 in. long and is tested in a vertical position with a 12,000-lb compressive load on the 4 by 4 in. top. The material is not straight grained but the grain makes an angle of 14° with the length. Find the unit shearing stress parallel to the grain and the unit compressive stress parallel to the grain. Work from fundamentals, finding the area and the component of the load tributary. *Ans.* $s_s = 176$ psi; $s_c = 706$ psi.

20-7. Solve Prob. 20-6 for a piece of timber which is of such poor quality that the grain makes an angle of 76° with the length.
Ans. $s_s = 176$ psi; $s_c = 43.9$ psi.

20-8. A 4- by 6- by 16-in. wood specimen carries a 10,000-lb compressive force on the 4- by 6-in. face. The grain of the material makes an angle of 20° with the length of the specimen. Find the shearing stress and the compressive stress parallel to the grain.

20-9. The grain of a 4- by 4-in. post of timber makes an angle of 12° with the length. Find the total safe load. Use the allowable working stresses of Tables 1 and 3.

21. Shearing Forces in Pairs. If a body is subjected to pure shearing stress (with no tension or compression except that which is due to

shear), there must be two sets of shearing forces to secure equilibrium and the unit shearing stresses which these forces produce must be the same in both. Figure 22 represents a rectangular block AB with two other blocks C and D glued to the top and bottom, respectively. There is a horizontal force P, toward the right, acting on the block C and an equal and opposite force acting on the block D. These two forces form a couple tending to rotate the system in a clockwise direction. To produce equilibrium, a block F is glued to the left vertical face of AB (Fig. 23) and a block G is glued to the right vertical face. A downward force Q is applied to F and an equal upward force is applied to G.

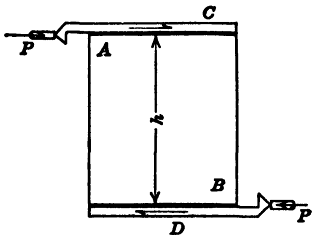


FIG. 22. Pair of shearing forces.

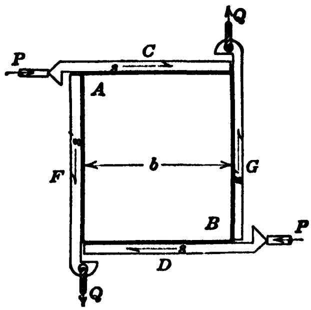


FIG. 23. Two pairs of shearing forces.

The breadth of AB is b and its height is h. Equilibrium will occur when the moments of the two couples are equal, *i.e.*, when

$$Ph = Qb \tag{21.1}$$

The force is transmitted from C and D to AB as a horizontal shear in the glue. Shearing stress is represented by an arrow with a single barb. The arrow in C, with barb above the shaft, represents the shearing stress from C to AB. If it were desired to represent the opposite shearing stress from AB to C, the arrow would be placed in AB, would point toward the left, and would have the barb below the shaft.

If l is the length of the block AB perpendicular to the plane of the paper, the top and bottom surfaces each have an area bl, and

$$P = sbl \tag{21.2}$$

in which s is the unit horizontal shearing stress.

The area of each vertical face perpendicular to the plane of the paper is hl and

$$Q = s'hl \tag{21.3}$$

in which s' is the unit vertical shearing stress.

Since

$$Ph = Qb$$
$$sblh = s'blh \qquad (21.4)$$
$$s = s' \qquad \text{Formula V}$$

Formula V applies to any portion of block *AB* cut out by horizontal and vertical planes perpendicular to the plane of the paper. Figure 24 represents one such block.

Frequently, tensile or compressive stresses occur along with shearing stresses. Figure 25 represents a block which is supposed to be glued to the base and pushed toward the right by a force *P* applied near the top. To the left of the middle the glue is in tension; to the right of the middle it is in compression. All the glue is in shear. A portion *A* of the block is in tension and shear, and a portion *C* is in compression and shear. The portion *B* at the middle is in shear only. The direction of the shear in *A* and *C*, for which the arrows are not shown, is the same as in *B*.

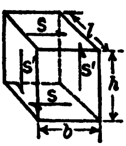


FIG. 24. Equilibrium in shear.

If the tension in *A* is not the same at the top and bottom, the vertical shearing stress will not be exactly equal on the two sides. Ordinarily, if *A* is small, the difference is slight.

For a block of infinitesimal dimensions, the shearing stresses are practically equal on all sides, even if tensile or compressive forces exist

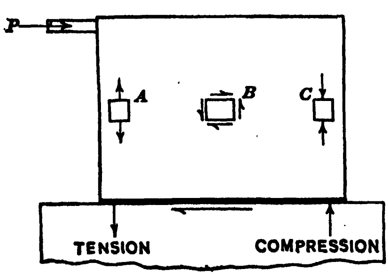


FIG. 25. Shear with tension and compression.

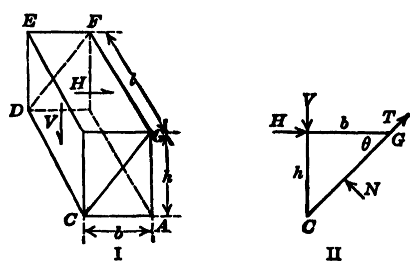


FIG. 26. Shear causing compression.

in the body. A combination of shear with other stresses is considered at greater length in Chap. 15.

22. Compressive and Tensile Stress Caused by Shear. Figure 26,I represents a rectangular parallelepiped of breadth *b*, height *h*, and length *l*, subjected to *pure* shearing stress. The shearing stress acts toward the right parallel to the breadth at the top and toward the left at the bottom. As shown in Art. 21, there is also a shearing stress of the same intensity at the left surface acting downward and an equal

shearing stress at the right surface acting upward. (If the direction of one of these shears is reversed, they must all be reversed to produce equilibrium.) Now consider the parallelepiped divided by the inclined plane containing the edges CD and GF, and treat the triangular prism to the left of this plane as a free body in equilibrium under the action of the forces at its surface. These forces are four in number: the shearing force H in the upper surface acting toward the right, the shearing force V in the left vertical surface acting downward, the compressive force N acting normal to the inclined surface (Fig. 26,II, which represents all the forces in the plane of the paper), and a shearing force T along this surface parallel to the diagonal line CG. If s_s is the intensity of the horizontal and vertical shear,

$$H = s_s bl \qquad V = s_s hl$$

Resolving normal to the inclined plane,

$$N = H \sin \theta + V \cos \theta \tag{22.1}$$

$$N = s_s bl \sin \theta + s_s hl \cos \theta \tag{22.2}$$

in which θ is the angle which the inclined plane makes with the horizontal surface. The unit normal stress on the inclined surface is obtained by dividing N by the area of this surface. If c is the length of the hypotenuse CG, the area of the inclined surface is cl. Dividing Eq. (22.2) by cl,

$$\frac{N}{\text{Area}} = \frac{N}{cl} = \frac{s_s b \sin \theta}{c} + \frac{s_s h \cos \theta}{c} \tag{22.3}$$

Since $\cos \theta = b/c$ and $\sin \theta = h/c$,

$$s_n = 2s_s \sin \theta \cos \theta = s_s \sin 2\theta \tag{22.4}$$

When θ is 45°, the normal stress is a maximum, is compressive, and is equal to the shearing stress.

$$s_n = s_c = s_s \qquad \text{Formula VI}$$

If the plane which bisects the parallelepiped is taken parallel to CD through the corners A and E, the same expression of Eq. (22.4) may be derived for the tensile stress. The maximum tensile stress is at 45° with the shearing stress and at right angles to the maximum compressive stress. When a body is subjected to pure shear, there is a compressive stress of equal intensity at an angle of 45° with the planes of the shearing stress in one direction and a tensile stress of the same intensity at an angle of 45° in the opposite direction. These are shown

in Fig. 27. With the shearing toward the left at the bottom, as indicated by the arrow, the maximum tensile stress is normal to the plane which makes an angle of 45° to the left of the vertical upward, and the maximum compressive stress is normal to the plane which makes an angle of 45° to the right of the vertical upward.

It is advisable now to consider the shearing force T which occurs on the plane CG in Fig. 26. Resolving parallel to the inclined plane,

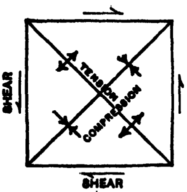


FIG. 27. Tension, compression, and shear.

$$T = H \cos \theta - V \sin \theta \tag{22.5}$$

Calling the shearing stress on the inclined plane s'_s,

$$T = s'_s cl = s_s bl \cos \theta - s_s hl \sin \theta \tag{22.6}$$

$$s'_s = s_s \cos^2 \theta - s_s \sin^2 \theta \tag{22.7}$$

$$s'_s = s_s \cos 2\theta \tag{22.8}$$

When θ is 0,

$$s'_s = s_s \tag{22.9}$$

which indicates that the maximum shearing stress occurs on the horizontal plane and is equal to s_s.

Problems

22-1. The unit shearing stress in a block is 200 psi and is directed toward the left at the bottom. Find the unit compressive stress across a plane which makes an angle of 25° with the horizontal toward the right.

Ans. 154 psi compressive.

22-2. In Prob. 22-1 find the unit shearing stress on the 25° plane. Draw a free body for the wedge and show the direction of this shearing stress.

Ans. 129 psi.

22-3. The force P of Fig. 16 causes a shearing stress of 250 psi on the element D. Find the shearing and normal stresses on a plane making an angle of 35° above the horizontal projecting to the right. Draw a free body of the wedge showing the directions of these forces.

22-4. Find the maximum and minimum shearing and normal stresses for Prob. 22-3, and show on your free body the planes on which these stresses occur.

22-5. Consider the block AB of Fig. 23 to be 15 in. wide, 8 in. high, and 1 in. thick. Forces P and Q cause 289-psi shearing stresses. Find the shearing and normal stresses on the diagonal AB. Work from fundamental principles, using the total forces and resolving.

22-6. Derive Eq. (22.4) by moments. Draw a free body similar to Fig. 26,II.

23. Graphical Solution for Normal and Shearing Stresses. A convenient method for finding the stresses on any plane when certain original stresses are given is Mohr's circle.[1] Let $s_x = P/A$, the original tensile stress applied on the area A in Fig. 28,I. Let s'_t and

[1] First developed by Otto Mohr about 1882. The methods proposed here are adapted to conform to some simplifications.

s'_s be the stresses on some plane making an angle θ with the original normal section. From Fig. 28,II, by resolutions,

$$s'_t = \frac{s_x}{2}(1 + \cos 2\theta) \tag{23.1}$$

$$s'_s = \frac{s_x}{2}\sin 2\theta \tag{23.2}$$

Plot (Fig. 28,III) on the positive X axis the tensile stress on the *left* face of the original (uncut) free body. If compression existed instead of tension, plot on the negative X axis the compressive stress. On the Y axis plot the shearing stress on the *left*-hand face, *in the direction in*

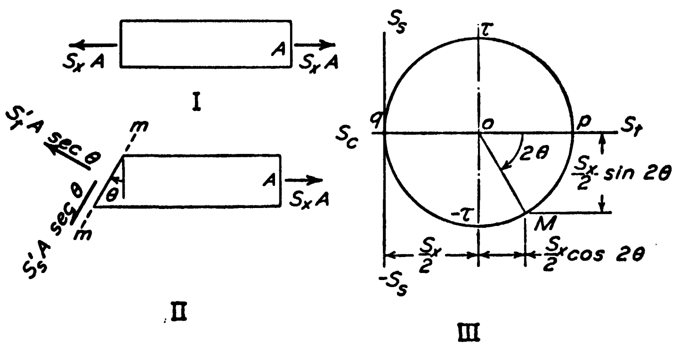


FIG. 28.

which it occurs. (There was no shearing stress in this case.) The tension on the top face is zero; hence plot a point at the origin and draw the circle. Figure 28,III is Mohr's circle from which the stresses are obtained. The maximum tensile stress is read at p and the minimum stress at q (at origin in this case). The maximum shearing stress is read at τ and the minimum at $-\tau$ (same value but opposite in sign and direction). The magnitude and direction of the normal and shearing stress on the plane mm are obtained from the point M. Note that the angle θ is measured from the normal section (where the stress s_x occurs) to the plane mm where stresses are desired. On Mohr's circle the corresponding angle is always 2θ and the direction of measuring 2θ is the same as for θ. The coordinates of M will be found to agree with Eqs. (23.1) and (23.2). The sign of the shear is negative, indicating that the direction of the shear in Fig. 28,II is down (on the left-hand side of the free body, of course).

Example 1

In Fig. 28 the applied unit tensile stress is 200 psi. Find the stresses on the plane where θ is 30°.

The maximum tensile stress will be 200 psi and is plotted as p. The minimum normal stress is zero and is plotted as q at the origin. Angle θ is measured clockwise, hence 2θ is 60° and is measured clockwise to M. From the trigonometry of the figure, the coordinates of M are found to be 150 psi tension and −86.6 psi shear. The minus sign indicates that the shearing stress is downward on the left face of the free body.

To find the maximum shear, the point M would move around to τ (Fig. 28,III) and 2θ would be 90° counterclockwise from the tension (horizontal toward the right) axis. Angle θ in the free body would be measured 45° counterclockwise from the normal section. The maximum shearing stress is the radius of the circle, 100 psi, and is directed upward on the left side of the free body. The minimum shear is numerically equal to the maximum shear. On what plane does it occur?

Problems

23-1. A body is subjected to a horizontal compressive stress of 1,000 psi. Draw Mohr's circle and find the stresses on a plane making an angle of 55° clockwise from the normal section. Find the maximum shear and indicate both planes where it occurs. (It is customary to indicate two planes of maximum shear instead of one minimum).

23-2. In Prob. 23-1 find the position of the plane where the stresses are $s_c' = 200$ psi and $s_s' = 400$ psi. Indicate your answer on your free body.

Ans. $\theta = 63°26'$ (clockwise).

23-3. In the example above find all the planes where the shearing stress has an absolute value of 80 psi. Find the normal stresses on these planes and draw free bodies to explain your answers.

Biaxial normal stresses will also develop shear on inclined planes. This case may also be solved graphically as follows:

For convenience on the free body of Fig. 29,II, let $s_x > s_y$. Taking a resolution parallel to mm,

$$s_s'A \sec\theta = s_xA \sin\theta - s_yA \tan\theta \cos\theta \tag{23.3}$$

$$s_s' = \frac{s_x - s_y}{2} \sin 2\theta \tag{23.4}$$

Similarly,

$$s_t' = \frac{s_x + s_y}{2} + \frac{s_x - s_y}{2} \cos 2\theta \tag{23.5}$$

Differentiating Eq. (23.4) will show that when $\theta = 45°$ or 135°,

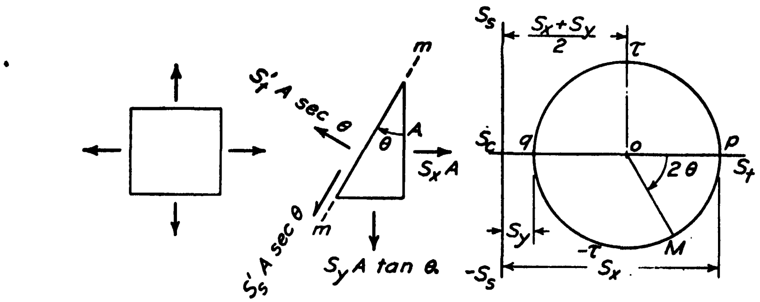
$$\text{Maximum shearing stress} = \tau = \frac{s_x - s_y}{2} \tag{23.6}$$

Likewise, from Eq. (23.5),

$$\text{Maximum tensile stress} = p = s_x \quad \text{when } \theta = 0 \tag{23.7}$$

$$\text{Minimum tensile stress} = q = s_y \quad \text{when } \theta = 90° \tag{23.8}$$

With these relations in mind, it is possible to plot s_x at p and s_y at q. With O as center, swing the circle, the radius of which is always τ, the maximum shearing stress. The stresses on any plane *mm* are found, as before, from the coordinates of M.



I II III

FIG. 29.

Example 2

In Fig. 29 the horizontal tensile stress is 800 psi and the vertical tensile stress is 200 psi. Find the stresses on the plane where $\theta = 35°$.

The maximum normal stress is 800 psi, which can be plotted as p. The minimum normal is 200 psi, which is q. The radius of the circle is $\tau = 300$ psi. When $\theta = 35°$ clockwise, 2θ is 70°, which locates M. By trigonometry

$$s_s' = 300 \sin 70° = 282 \text{ psi.}$$

This is negative and hence downward as in Fig. 29,II. Also

$$s_t' = 500 + 300 \cos 70° = 503 \text{ psi.}$$

The maximum shearing stresses occur on planes making angles of $\pm 45°$ with the plane on which the 800-psi stress acts. (On which plane is the shear up?)

Problems

23-4. If $s_x = 700$ psi and $s_y = 300$ psi, find p, q, and τ and illustrate by free-body diagrams showing the planes. *Ans.* 700 psi; 300 psi; 200 psi.

23-5. If $s_x = 700$ psi and $s_y = 300$ psi, find the stresses on the plane where $\theta = 40°$ measured as in Fig. 29,II. Show the directions on your free bodies. *Ans.* 535 psi; -197 psi.

23-6. Solve Prob. 23-5 for an angle $\theta = 80°$.

23-7. Solve Prob. 23-4 if $s_y = -300$ psi. *Ans.* 700 psi; -300 psi; 500 psi.

23-8. Solve Prob. 23-5 if $s_y = -300$ psi. *Ans.* 287 psi; -492 psi.

23-9. Solve Prob. 23-6 if $s_y = -300$ psi.

23-10. In Prob. 23-7, on which planes will there be no normal stress? Draw free bodies to illustrate.

Pure shear will also develop shear and normal stresses on inclined planes. This case may be solved graphically as follows:

Since the unit horizontal shearing stress equals the vertical shearing stress, from Formula V, the element of Fig. 30,I is cut by a plane *mm* in Fig. 30,II and a resolution taken parallel to the plane.

$$s_s'A \sec\theta = s_sA\cos\theta - s_sA\tan\theta\sin\theta \tag{23.9}$$

$$s_s' = s_s\cos 2\theta \tag{23.10}$$

Similarly,

$$s_t' = s_s\sin 2\theta \tag{23.11}$$

Differentiating and substituting,

$$\tau = s_s \quad \text{when } \theta \text{ is } 0° \text{ or } 90° \tag{23.12}$$

$$p = s_s \quad \text{when } \theta = 45° \tag{23.13}$$

$$q = -s_s \quad \text{when } \theta = ? \tag{23.14}$$

Mohr's circle is shown in Fig. 30,III.

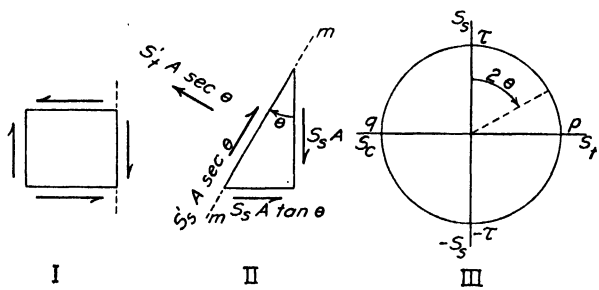


FIG. 30.

Example 3

The unit shearing stress at a certain point on a given element is 300 psi and upward on the left side of the element as shown in Fig. 30,I. Find the maximum stresses and the stresses on a 20° plane clockwise.

When one shearing force is given, the other three must satisfy the conditions of equilibrium. On the left-hand face there is no normal stress; hence the coordinates to be plotted are (0, 300) in Fig. 30,III. When the shear forces form a clockwise couple, the shear is plotted as positive. The shear on the top and bottom faces for a counterclockwise couple is negative and is plotted as (0, −300) since there is no normal stress on these faces. On the free body, the stresses on any two planes which are 90° apart are indicated by the coordinates of two points which are 180° apart on the circle. Hence the center of the circle is at the origin, and $p = 300$ psi, $q = -300$ psi. Since $\theta = 20°$, $2\theta = 40°$ clockwise and the coordinates of M are $s_t' = 193$ psi and $s_s' = 230$ psi. The directions are indicated on Fig. 30,II.

Problems

23-11. In the example above, find the stresses when the 20° angle is changed to 70°.

23-12. If the applied unit shearing stress is 200 psi and is directed downward on the

left side of the element, find the stresses on a plane where $\theta = 60°$ counterclockwise from the vertical.

Ans. $s_s = 100$ psi; $s_n = 173$ psi (tension or compression?).

23-13. Solve Prob. 23-12 for $\theta = 18°$ counterclockwise.

23-14. Derive formulas for s_s' and s_t' in terms of ϕ, the angle measured from the plane of maximum tensile stress, p. Hence show that this case may be solved as a problem of biaxial stress.

24. Energy of Shear. When shearing stresses are applied to an element, as in Fig. 30,I, the work done can be calculated from the average stress times the unit shearing deformation. If the bottom of the element is assumed fixed while the top edge is displaced a small amount to the left, the energy is

$$u_s = \frac{s_s}{2}\gamma = \frac{s_s}{2} \times \frac{s_s}{G} = \frac{s_s^2}{2G} \tag{24.1}$$

since the vertical shears do no work during the small distortion.

Problems

24-1. If the shearing elastic limit of a steel is 24,000 psi, find the unit energy stored up when stressed to this value. *Ans.* 24.0 in.-lb per cu in.

24-2. The applied unit shearing stresses on an aluminum element are 8,000 psi. Find the unit energy absorbed.

25. Relation of E to G and μ. Consider the element of Fig. 31, which is subjected to pure shear. It was proved in Eqs. (23.13) and (23.14) that the maximum and minimum normal stresses which occur when the shearing stresses are applied are

$$p = s_s \tag{25.1}$$
$$q = -s_s \tag{25.2}$$

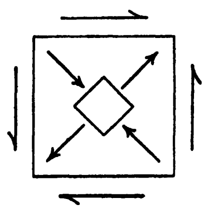


FIG. 31.

and that these maximum tensile and compressive stresses occur at an angle of 45° with the original applied shear. The energy stored up per cubic inch will be

$$u = \frac{1}{2E}(p^2 + q^2 - 2\mu pq) \tag{25.3}$$

The energy stored up by the original shearing stresses is

$$u = \frac{s_s^2}{2G} \tag{25.4}$$

Since the energy must be the same from either equation[1],

[1] This original solution was first suggested to the author in 1939 by his colleague, Prof. C. T. West, then a graduate student.

$$\frac{s_s^2}{2G} = \frac{1}{2E}(s_s^2 + s_s^2 + 2\mu s_s^2) \tag{25.5}$$

Hence

$$G = \frac{E}{2(1 + \mu)} \qquad \text{Formula VII}$$

Problems

25-1. If Poisson's ratio is 1/4 show that $G = 2E/5$.

25-2. If the modulus of elasticity of steel is 29,800,000 psi and Poisson's ratio is 0.275, what is the modulus of rigidity? *Ans.* $G = 11{,}690{,}000$ psi.

25-3. Find Poisson's ratio if the modulus of rigidity is 11,500,000 and the modulus of elasticity in tension is 29,300,000 psi.

25-4. If Poisson's ratio for aluminum is 1/3, find the modulus of rigidity. *Ans.* $G = 3{,}975{,}000$ psi.

25-5. For brass (34 % zinc), the modulus of elasticity is 15,000,000 psi and Poisson's ratio is 1/3. Find the modulus of rigidity.

25-6. For zinc the modulus of elasticity is 12,000,000 psi and Poisson's ratio is 0.11. Find the modulus of rigidity.

25-7. If the modulus of elasticity for Monel metal is 25,000,000 psi and Poisson's ratio is 0.39, what is the modulus of rigidity?

25-8. If Poisson's ratio for aluminum is 0.33 and the modulus of elasticity varies from 10,000,000 to 11,000,000 psi for different alloys, how much does the modulus of rigidity vary?

26. Miscellaneous Problems

26-1. Figure 32 shows a block which is subjected to horizontal tension and vertical compression. If the unit tensile stress is 240 psi and the unit compressive stress is 200 psi, what is the unit shearing stress along the diagonal? *Ans.* $s_s = 220$ psi.

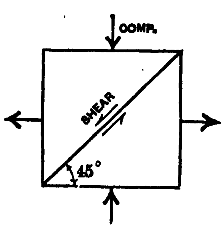


FIG. 32. Shear caused by tension and compression.

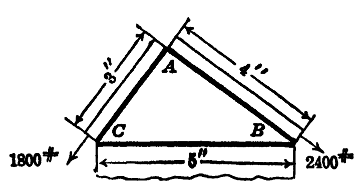


FIG. 33. Stress due to shear.

26-2. A rectangular block is 15 in. wide, 8 in. high, and 4 in. long. A horizontal compression force of 27,744 lb in the direction of its width is applied to the sides. A vertical tension of 34,680 lb is applied to the top and bottom surfaces. Find the unit shearing stress along a plane which passes through the lower left edge and the upper right edge. Solve without using a formula. *Ans.* $s_s = 600$ psi.

26-3. In Fig. 33 the block ABC is 8 in. long perpendicular to the plane of the paper. Find the unit shearing stress and the unit compressive stress in the glue which fastens the block to its base.

26-4. In Fig. 34, A and B are short compression members or struts of timber joined together by a bolt or pin at the top. The lower ends are set in notches in the bottom chord C. If the load P is 7,200 lb, what is the unit compressive stress in A and B? What is the maximum unit tensile stress in C? What must be the length of the section d to avoid shearing if C is made of timber?

26-5. What must be the thickness t of the supports of Fig. 34 if C is timber? If C is oak?

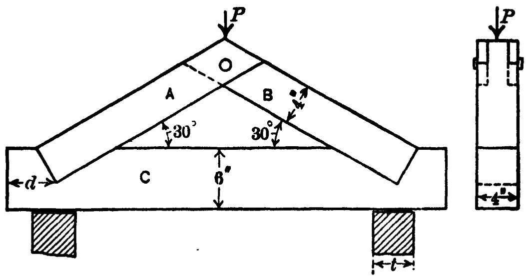


FIG. 34. Stresses in a truss.

26-6. A horizontal beam is 5 ft long and is hinged at the left end A. The beam carries a load of 1,800 lb 3 ft from A and is supported by a steel rod at the right end B. The rod is attached to a point which is 6 ft above the hinge. What should be the minimum diameter of the rod?

26-7. The force P in Fig. 16 causes a shearing stress of 350 psi on the element D. Using Mohr's circle, find the unit shearing and normal stress on a plane making an angle of 20° with the horizontal to the left. Show the forces on your free body.

26-8. The A frame shown in Fig. 35 is made of three members which are fastened together with unfinished steel bolts at C, D, and E. Find the diameter to be used at D.

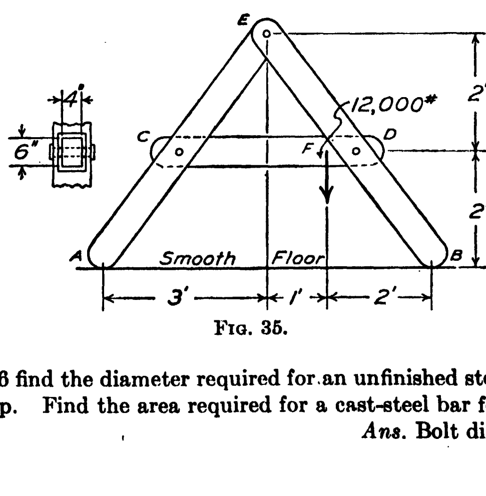


FIG. 35.

26-9. In Fig. 36 find the diameter required for an unfinished steel bolt to be used at the top. Find the area required for a cast-steel bar for the link.

Ans. Bolt diameter = 1.66 in.

26-10. In Fig. 37 the walls at A and B are 160 in. apart and remain fixed in position. The four bars AC, AD, BC, and BD are 100 in. long and $\frac{3}{8}$ in. in diameter. The pins at A and B are in single shear. The jack has three threads to the

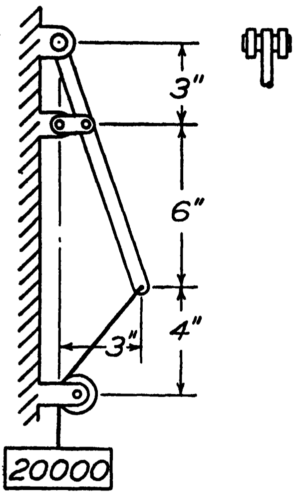


Fig. 36.

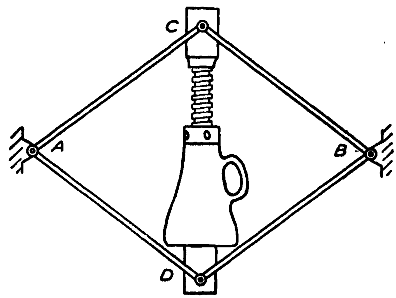


Fig. 37.

inch and is large enough so that the deformation in it may be neglected. When a quarter turn is taken on the jack find (a) the size of the pins A and B and (b) elongation and stress in the nickel-steel bars.

CHAPTER 3

STRESS BEYOND THE ELASTIC LIMIT

27. Ultimate Strength. The *ultimate strength* of any material is the maximum unit stress which it can exert. In this sense *ultimate* means the greatest, which is not necessarily the last load before failure. In most testing machines, the load is applied by a screw or by a hydraulic press. When the maximum load is reached, the material is deformed considerably. Since the application of the load is not instantaneous, the stress drops. The final or breaking load may be much less than the ultimate. For instance, a steel rod of 1 square inch cross section may have an ultimate strength of 58,000 pounds. If an actual weight were hung on this rod and additional loads were added until 58,000 pounds was reached, the load would remain 58,000 pounds, no matter what the deformation might be. With a load of this kind, which follows up the deformation, the ultimate load is the last load. On a testing machine, the maximum load is read at 58,000 pounds. The machine then slowly elongates the material at smaller stress. Just before the rupture, the load may read 40,000 pounds. This is called the *breaking load.*

28. Stress-Strain Diagrams for Timber. Stresses below the elastic limit are considered in Chap. 1. Below that limit unit stress is proportional to unit deformation; Formula III of Art. 9 and the equations of resilience and change of volume hold good. Unit stress below the elastic limit is most important from the standpoint of the engineer, for allowable stresses in correctly designed structures are usually kept below that limit.

It is desirable, however, to know what takes place above the elastic limit, and to understand the conditions of complete failure for the various structural materials. To gain this knowledge, experiments are made in which a series of loads are applied to a test piece of the material, and the corresponding deformations are observed with suitable measuring apparatus.

Table 4 gives the results of a compression test of a stick of longleaf yellow pine. Loads were applied and the resistance weighed by means of a 50,000-pound testing machine. The deformation in an 8-inch

TABLE 4. COMPRESSION TEST OF LONGLEAF YELLOW PINE

Length, 12.5 inches; cross section 1.62 inches by 1.48 inches = 2.40 square inches. Weight of piece, 12.25 ounces. Gage length, 8 inches. Lever extensometer magnifies five times; dial readings, 0.0001 inch.

Total load, pounds	Unit stress, per square inch	Dial reading, $\frac{1}{50,000}$ inch	Compression		
			In gage length		Unit, inches per inch
			$\frac{1}{50,000}$ inch	Inches	
10	4	4,728	0	0	0
480	200	4,700	28	0.00056	0.000070
960	400	4,669	59	118	147
1,440	600	4,638	90	180	225
1,920	800	4,607	121	242	302
2,400	1,000	4,575	153	0.00306	0.000382
2,880	1,200	4,546	182	364	455
3,360	1,400	4,518	210	420	525
3,840	1,600	4,488	240	480	600
4,320	1,800	4,459	269	538	672
4,800	2,000	4,426	302	0.00604	0.000755
5,280	2,200	4,398	330	660	825
5,760	2,400	4,368	360	720	900
6,240	2,600	4,338	390	780	975
6,720	2,800	4,309	419	838	0.001047
7,200	3,000	4,278	450	0.00900	0.001125
7,680	3,200	4,248	480	960	1200
8,160	3,400	4,216	512	0.01024	1280
8,640	3,600	4,187	541	1082	1352
9,120	3,800	4,157	571	1142	1427
9,600	4,000	4,122	606	0.01212	0.001515
10,080	4,200	4,089	639	1278	1597
10,560	4,400	4,058	670	1340	1675
11,040	4,600	4,026	702	1404	1755
11,520	4,300	3,996	732	1464	1830
12,000	5,000	3,960	768	0.01536	0.001920
12,480	5,200	3,928	800	1600	2000
12,960	5,400	3,897	831	1662	2077
13,440	5,600	3,866	862	1724	2155
13,920	5,800	3,830	898	1796	2245
14,400	6,000	3,799	929	0.01858	0.002322
14,880	6,200	3,760	968	1936	2420
15,360	6,400	3,731	997	1994	2492
15,840	6,600	3,700	1,028	2056	2570
16,320	6,800	3,668	1,060	2120	2650
16,800	7,000	3,628	1,100	0.02200	0.002750
17,280	7,200	3,589	1,139	2278	2847
17,760	7,400	3,556	1,172	2344	2930
18,290	7,622	3,515	1,213	2426	3032
18,720	7,800	3,478	1,250	2500	3125
19,200	8,000	3,439	1,289	0.02578	0.003222
19,680	8,200	3,400	1,328	2656	3320
20,160	8,400	3,362	1,366	2732	3415
20,700	8,625	3,310	1,418	2836	3545
	Extensometer removed. Beam kept in balance				
26,650	11,104				
27,050	11,271				
27,150	11,312	Ultimate strength			
26,700	11,125				

gage length was measured by a lever micrometer with an arm ratio of 1:5. The magnified deformation was read on an Ames dial graduated to 1/1,000 inch for each division. By estimating tenths of a division the readings become 1/10,000 inch on the dial, which corresponds to a deformation of 1/50,000 inch for the gage length of the test piece.[1]

The first column of Table 4 gives the total load. The second column gives the unit stress. *Unit stress* is the most important quantity which should be kept in mind. It is customary to apply total loads which give convenient equal increments of unit stress. In this test the increment was 200 pounds per square inch.

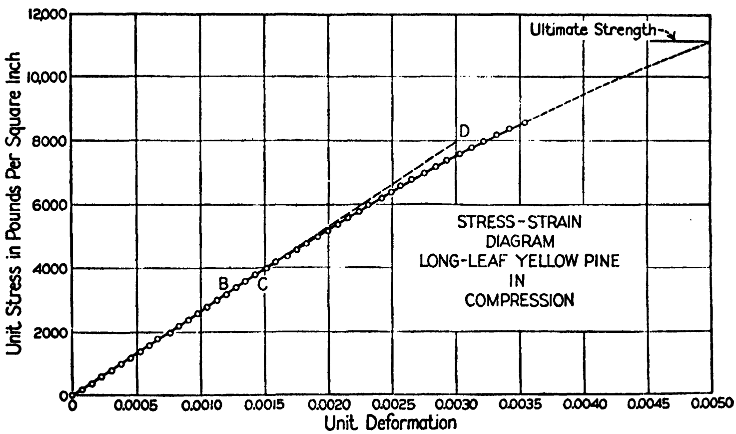


FIG. 38. Stress-strain diagram for Table 4.

The third column gives the dial reading, each integer corresponding to a deformation of 0.00002 inch in the 8-inch gage length. The fourth column gives the deformation obtained by subtracting the dial reading at the initial 10-pound load from each of the others. The fifth column gives the deformation in inches. The last column is the unit deformation.

Figure 38 is the *stress-strain diagram* for Table 4. (It is customary in America to plot strain horizontally and stress vertically. Some British writers take stress horizontally and strain vertically.)

When the data have been plotted on cross-section paper, it is found that the points up to a stress of nearly 4,000 pounds per square inch lie approximately on a straight line. From thousands of tests it has been shown that the stress-strain diagrams of the best elastic materials are

[1] This extensometer reads the same as the well-known Berry strain gage, except that increased reading means positive elongation.

smooth curves which approximate straight lines for a considerable range (Hooke's law).[1]

The location of the straight line of Fig. 38 after the points have been plotted is comparatively easy. A straight line through the origin of coordinates and the point B at which the unit stress is 3,200 pounds per square inch is found to pass directly through seven of the intervening points. The straight line to B is extended as a broken line from C to D (the portion from B to C omitted to avoid crowding) and a smooth curve is drawn beyond B. The stress of 3,200 pounds may be regarded as the point of tangency at which the curve leaves the straight line. However, a straight line of slightly different slope might have been drawn from a point a little above the origin through a point just to the right of the 3,800-pound circle. The point of tangency of this straight line to the curve would lie at about 3,800 pounds per square inch.

The point of tangency, at which the curved portion of the stress-drain diagram leaves the straight line, is called the *proportional elastic limit.* Below the proportional elastic limit the equations

$$E = \frac{s}{\epsilon} \qquad \text{and} \qquad U_2 - U_1 = \frac{s_2^2 - s_1^2}{2E}$$

apply. Above the proportional elastic limit these equations are no longer valid.

The *elastic limit* was defined in Art. 8 as the maximum stress which may be applied without *permanent set.* This limit is difficult to locate, especially when very precise measurements are made. It should be understood that the definition of elastic limit implies a repeated loading and unloading of the specimen as was done for the timber of Table 6. Roughly the elastic limit and the proportional limit are nearly the same, and for practical reasons the latter is almost always obtained. Several other points on the stress-strain diagrams are often substituted for the elastic limit. *Johnson's apparent elastic limit* is the stress where the rate of deformation is 50 per cent greater than at the origin (see Art. 30). *Yield point* refers to a stress where the deformation in a ductile material continues to increase without an increase in stress (see Art. 29). The *yield strength* is an invention to describe the stress where the permanent set is $\frac{2}{10}$ of 1 per cent (see Art. 30). None of these are really elastic limits.

Table 5 gives the results of a compression test of Douglas fir, which

[1] Sir Robert Hooke (1635–1702) announced in 1678 the proportionality between stress and strain, but engineers overlooked its importance for a century.

TABLE 5. COMPRESSION TEST OF DOUGLAS FIR, LENGTHWISE OF THE GRAIN*

Length, 59.6 inches; cross section, 4.1 inches by 11.96 inches = 49.04 square inches. Gage length, 50 inches. Weight, 39.8 pounds per cubic foot.

Total load, pounds	Unit stress, pounds per square inch	Total compression, inches	Compression for 200-lb. stress, inches	Unit compression, inches per inch
4,904	100	0	......	0
9,808	200	0.0021	......	0.000042
14,712	300	44	0.0044	88
19,616	400	68	47	0.000136
24,520	500	92	48	184
29,424	600	0.0117	0.0049	0.000234
34,328	700	141	49	282
39,232	800	166	49	332
44,136	900	191	50	382
49,040	1,000	216	50	432
58,848	1,200	0.0261	0.0045	0.000522
68,656	1,400	314	53	628
78,464	1,600	364	50	728
88,272	1,800	415	51	830
98,080	2,000	466	51	932
107,888	2,200	0.0514	0.0048	0.001028
117,696	2,400	568	54	1136
127,504	2,600	620	52	1240
137,312	2,800	674	54	1348
147,120	3,000	726	52	1452
4,904	100	0.0004	......	0.000008
305,050	6,220 Ultimate strength			

Failed by triple flexure. Fibers crushed 12 inches from one end of stick.

* Table 5 is taken from "Tests of Metals," 1896, p. 415.

was made at the Watertown Arsenal. The initial load was 100 pounds per square inch. It is customary, generally, to start the deformation readings at some conveniently low stress rather than at zero. (The machine at Watertown Arsenal is horizontal.)

The results of Table 5 are plotted as Fig. 39 parallel to the grain. Inspection shows that the best straight line intersects zero deformation at unit stress of about 140 pounds per square inch, instead of 100 pounds per square inch, which is the location of the point representing the initial load. On account of "lost motion" in the extensometers

or residual deformations in the test piece, one or more deformations at the beginning are frequently off the straight line. The elastic limit may be taken as somewhere between 2,600 and 2,800 pounds per square inch.

Table 6 is the record of a compression test on a block of Douglas fir. The pressure was applied across the grain in a direction perpendicular to the growth rings. The initial load was 20 pounds per square inch. At every hundred pounds the load was reduced to the initial 20 pounds per square inch. The set was found to be very small up to

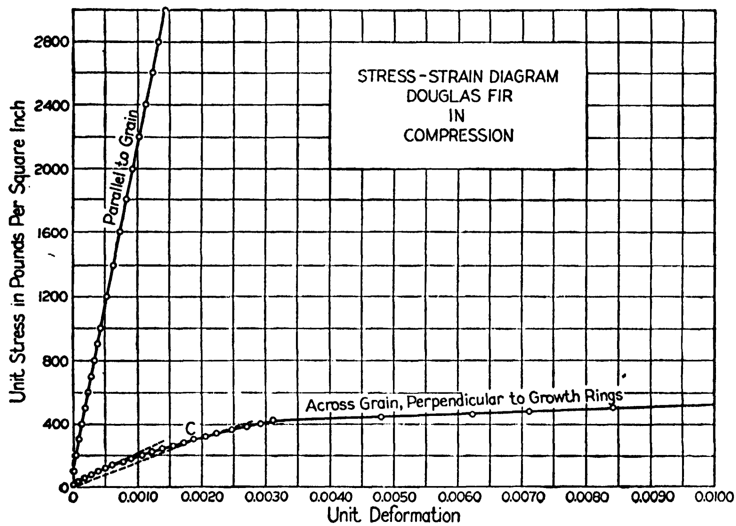


Fig. 39. Stress-strain diagrams for Tables 5 and 6.

400 pounds per square inch but became very large with increasing loads.

The data of Table 6 up to a stress of 500 pounds per square inch are plotted as Fig. 39 (across the grain). Apparently the proportional elastic limit lies between 140 and 160 pounds per square inch. The measurements show zero set at 100 pounds per square inch and a very small set at 200 pounds per square inch. These seem to indicate an elastic limit between 100 and 200 pounds. This fulfills the definition of elastic limit as given in Art. 8.

Since instrumental errors, internal stresses, and inaccuracy of the experimenter combine to give results which do not fall exactly on a mathematically straight line, practical methods must be devised for getting a fairly correct value of the modulus of elasticity without too much labor. The slope of the stress-strain diagram affords one

TABLE 6. COMPRESSION TEST OF DOUGLAS FIR ACROSS THE GRAIN, PERPENDICULAR TO GROWTH RINGS (FIG. 29, No. 1430)*

Length, 8.11 inches; cross section, 9.63 inches by 9.69 inches. Area perpendicular to 9.69-inch dimension, 78.1 square inches. Gage length, 6 inches.

Total load, pounds	Unit stress, pounds per square inch	Total compression, inches	Unit compression, inches per inch
1,562	20	0	0
3,124	40	0.0004	0.00007
4,686	60	0.0011	0.00018
6,248	80	16	27
7,810	100	23	38
1,562	20	0 set	0
9,372	120	0.0030	0.00050
10,934	140	37	62
12,496	160	47	78
14,058	180	54	90
15,620	200	64	0.00107
1,562	20	0.0001 set	0.00002
17,182	220	0.0073	0.00122
18,744	240	83	138
20,306	260	93	155
21,868	280	0.0103	172
23,430	300	113	188
1,562	20	0.0004 set	0.00007
24,992	320	0.0123	0.00205
26,554	340	133	222
28,116	360	149	248
29,678	380	162	270
31,240	400	175	292
1,562	20	0.0010 set	0.00017
32,802	420	0.0187	0.00312
34,363	440	288	480
35,926	460	373	622
37,488	480	426	710
39,050	500	504	840
1,562	20	0.0219 set	365
40,612	520	0.0690	0.01150
42,174	540	0.13	0.022
43,736	560	0.28	47
45,298	580	0.42	70
46,860	600	0.60	0.100
54,670	700	1.24	0.207
62,480	800	1.80	0.300
70,290	900	2.03	0.338
78,100	1,000	2.24	0.373
156,200	2,000	2.75	0.458
234,300	3,000	2.94	0.490

Fibers crushed laterally, but wood did not split along the grain.

* From "Tests of Metals," Watertown Arsenal, 1896, p. 389.

method. The straight line of Fig. 39 for Douglas fir, parallel to the grain, intersects zero deformation at unit stress of about 140 pounds per square inch and passes through the line of 0.001 deformation at about 2,140 pounds per square inch.

$$E = \frac{2{,}140 - 140}{0.001 - 0} = \frac{2{,}000}{0.001} = 2{,}000{,}000 \text{ psi}$$

For Douglas fir in compression across the grain, perpendicular to the growth rings, the curve of Fig. 39 gives

$$E = \frac{210 - 35}{0.001 - 0} = 175{,}000 \text{ psi}$$

Modulus of elasticity calculated from the slope of the straight line may be regarded as an approximate average, the accuracy of which depends upon the exactness with which the straight line has been located with reference to the experimental points.

The modulus of elasticity may be computed directly from the experimental table. While the plotted curve is not absolutely necessary, it should be drawn to give the limiting values of the readings that may be used. To get an average result, several equal intervals should be calculated. Since the errors in balancing the scale beam and reading the extensometer are practically constant, the *relative* errors are proportional to the length of the interval. To obtain the maximum accuracy, the intervals of unit stress should be made as large as possible. No interval should extend above the elastic limit. If one or more points at the lower end of the line are distinctly off the line in the *same direction*, there points should not be used.

Example

Find the modulus of elasticity of the longleaf yellow-pine stick of Table 4 from four intervals of 2,000 lb each, beginning with the interval from 200 to 2,200 lb.

Stress interval, psi	Unit-deformation interval	Modulus of elasticity, psi
200 to 2,200	0.000825 − 0.000070 = 0.000755	2,649,000
400 to 2,400	0.000900 − 0.000147 = 0.000753	2,656,000
600 to 2,600	0.000975 − 0.000225 = 0.000750	2,667,000
800 to 2,800	0.001047 − 0.000302 = 0.000745	2,684,000
		256 ÷ 4 = 64
		Average 2,664,000

Instead of calculating E for each interval and averaging the results, the average value of the deformation intervals might be calculated, and a single value of E computed from this average. It is best, however, to calculate each modulus separately in order to see how great is the variation in a single test and to be able to compare the relative accuracy of different tests in terms of the desired quantity. The labor of division is negligible if a table of reciprocals is used.

Problems

28-1. Find the modulus of elasticity of the Douglas fir of Table 5 from four intervals of 1,000-lb unit stress, beginning with the interval from 200 to 1,200 lb. Use total deformation intervals from the third column of the table. Average these and find the average unit deformation.
Ans. $E = 1{,}000 \div 0.00049 = 2{,}041{,}000$ psi.

28-2. Solve Prob. 28-1 from four intervals of 1,000-lb unit stress beginning with the interval from 1,200 to 2,200 lb and ending with the interval from 1,800 to 2,800 lb. *Ans.* $E = 1{,}000 \div 0.00051 = 1{,}961{,}000$ psi.

28-3. Find E for Douglas fir across the grain, perpendicular to the growth rings, from Table 6. Use three intervals of 60 lb, beginning with the interval from 40 to 100 lb.

28-4. Find E for the longleaf yellow pine of Table 4, using four intervals of 1,000 lb, beginning with 400 psi, and four intervals of 1,000 lb ending with 3,200 psi. Solve as in Prob. 28-1. *Ans.* 2,676,000 psi; 2,676,000 psi.

28-5. A yellow-pine specimen was 1.66 by 1.68 in., and 9.96 in. long. The maximum load in compression was 17,000 lb. Find the ultimate strength.

28-6. A second specimen similar to Prob. 28-5 was 1.75 by 1.78 in. and carried a maximum load of 25,200 lb. Find the ultimate strength.

28-7. A block of Douglas fir was 9.64 in. wide by 8.07 in. thick and 9.64 in. high. When the total load on top changed from 3,128 to 14,076 lb, the gage reading in a 6-in. gage length changed from 0.0014 to 0.0140 in. The loads were applied across the grain and the maximum load was 54,740 lb. Find the ultimate strength and the modulus of elasticity across the grain.
Ans. $E = 66{,}700$ psi.

29. Stress-Strain Diagrams for Steel. Table 7 gives the results of the tension test of a rod of low-carbon steel. Figure 40,I shows the entire stress-strain diagram. Figure 40,II gives a small portion of the diagram with the horizontal scale 100 times as great as that of Fig. 40,I. The elastic limit is located at B at a unit stress of 35,000 pounds per square inch. From C to C_1 there is a relatively large elongation with very little increase of stress. From C_1 to C_2 the unit stress drops from 36,520 to 36,490 pounds per square inch (see Table 7). The point at which the deformation increases with no increase of stress is called the *yield point* of the material. The yield-point stress of Table 7 may be taken at 37,940 pounds per square inch.

Not all of a test bar reaches the yield point at the same time. A small portion yields; the stress drops and then gradually rises. Another portion yields and there is another sudden drop. With the

TABLE 7. TENSION TEST OF LOW-CARBON STEEL

Gage length, 8 inches; mean diameter, 0.5993 inch; area, 0.28208 square inch; approximate area, 0.282 square inch. The crosshead speed of the testing machine was $\frac{1}{60}$ inch per minute.

Total load, pounds	Unit stress, pounds per square inch	Dial reading, 0.00002 inch	Elongation		
			In gage length		Unit, inches per inch
			0.00002 inch	Inches	
0	0	16	0	0	0
282	1,000	30	14	0.00028	0.000035
564	2,000	40	24	48	60
846	3,000	52	36	72	90
1,128	4,000	66	50	0.00100	0.000125
1,692	6,000	93	77	0.00154	0.000192
2,256	8,000	120	104	208	260
2,820	10,000	146	130	260	325
3,384	12,000	175	159	0.00318	0.000397
3,948	14,000	200	184	368	460
4,512	16,000	229	213	426	532
5,076	18,000	256	240	480	600
5,640	20,000	282	266	532	665
6,204	22,000	309	293	0.00586	0.000732
6,768	24,000	336	320	640	800
7,332	26,000	364	348	696	870
7,896	28,000	391	375	750	937
8,640	30,000	427	411	822	0.001027
8,742	31,000	434	418	0.00836	0.001045
9,024	32,000	448	432	864	1080
9,306	33,000	456	440	880	1100
9,588	34,000	467	451	902	1127
9,870	35,000	482	466	932	1165
10,000	35,460	491	475	0.00950	0.001187
10,200	36,170	502	486	972	1215
10,300	36,520	607	591	0.01182	1477
10,290	36,490	665	649	1298	1622
10,540	37,380	780	764	1528	1910
10,530	37,340	890	874	0.01748	0.002185
10,530	37,340	980	964	1928	2410
10,640	37,730	1,127	1,111	2222	2777
10,450	37,060	1,431	1,415	2830	3537
10,610	37,624	1,553	1,537	3074	3842
10,500	37,230	1,750	1,734	0.03468	0.004335
10,630	37,700	2,080	2,064	4928	5160
10,600	37,590	2,480	2,464	4928	6160
10,410	36,910	2,945	2,929	5858	7322
10,580	37,520	3,310	3,294	6588	8235
10,460	37,090	3,810	3,796	0.07592	0.009490
10,700	37,940	4,070	4,054	8108	0.010135
10,440	37,020	4,360	4,344	8688	10860
10,560	37,450	4,670	4,654	9308	11635
10,520	37,310	4,740	4,724		
		4,765	4,749		
	After 30 sec. reset to 376 new zero = 376 − 4,749 = −4,373				
10,400	36,880	695	5,068	0.10136	0.012670
10,500	37,230	960	5,333	.10666	13442
10,450	37,060	1,227	5,600	.11200	16345
10,630	37,700	2,165	6,538	.13076	16345
10,680	37,870	2,755	7,128	.14256	17820
10,550	37,410	3,160	7,533	0.15066	0.018882
10,440	37,020	4,060	8,433	.16866	21082
10,300	36,520	4,695	9,068	.18136	22670
		4,730	9,103		

TABLE 7. TENSION TEST OF LOW-CARBON STEEL.—(*Continued*)

Total load, pounds	Unit stress, pounds per square inch	Dial reading, 0.00002 inch	Elongation: In gage length, 0.00002 inch	Elongation: In gage length, Inches	Elongation: Unit, inches per inch
After standing reset to 600 new zero = −8,503					
10,550	37,410	1,170	9,678	0.19356	24182
10,720	38,010	1,270	9,773	.19546	24432
10,990	38,970	1,550	10,053	0.20106	0.025132
11,100	39,360	1,850	10,353	.20706	25882
11,200	39,720	2,081	10,584	.21168	26435
11,300	40,070	2,370	10,873	.21746	27182
11,460	40,640	2,850	11,353	.22706	28382
11,590	41,100	3,300	11,803	0.23606	0.029507
11,700	41,490	3,550	12,053	.24106	30132
11,850	42,020	4,120	12,623	.25246	31557
11,940	42,340	4,507	13,010	.26020	32525

Removed extensometer. Reading with dividers and scale at 12,000 pounds load was 0.26 inch.

Total load, pounds	Unit stress, pounds per square inch	Elongation: In 8 inches	Elongation: Unit	
12,180	43,190	0.30	0.0375	*Machine run at 1 in. per minute for about 0.08 in. Then run at $\frac{1}{60}$ in. per minute for the remaining 0.02 in. of each interval*
13,150	46,630	.40	500	
14,000	49,650	.50	625	
14,570	51,670	.60	750	
14,970	53,080	.70	975	
15,330	54,360	0.80	0.1000	
15,600	55,320	.90	.1125	
15,790	55,990	1.00	.1250	
15,940	56,250	1.10	.1375	
16,050	56,920	1.20	.1500	
16,100	57,090	1.30	0.1625	
16,220	57,520	1.40	.1750	
16,270	57,700	1.50	.1875	
16,315	57,850	1.60	.2000	
16,330	57,910	1.70	.2125	
16,345	57,960	1.80	0.2250	
16,365	58,030	1.90	.2375	
16,350	57,980	2.00	.2500	
16,350	57,980	2.10	.2625	
16,330	57,910	2.20	.2750	
16,280	57,730	2.30	0.2875	
16,200	57,450	2.40	.3000	
16,120	57,160	2.50	.3125	
16,150	57,270	2.55	.3187	Diameter of neck 0.483 in
15,720	55,740	2.60	.3250	

Total load, pounds	Unit stress, pounds per square inch	Elongation: In 8 inches, inches	Elongation: Unit, inches per inch	Diameter of neck, inches	
15,300	54,260	2.65	0.3312	0.457	Ran machine entirely at $\frac{1}{60}$ in. per minute after necking began
14,580	51,702	2.68	.3350	.441	
14,150	50,180	2.70	.3375	.428	
13,800	48,940	2.72	.3400	.413	
13,130	46,560	2.74	.3425	.395	
12,700	45,040	2.75	0.3432	0.385	
11,520	40,850	2.76	.3450	.339	

testing machine elongating the specimen with a constant slow speed and the beam balanced, there is a sudden "drop of the beam" when a new portion reaches the yield point. Before the beam can be balanced again, there is a considerable elongation; the curves, therefore, fail to show the true abruptness of reduction of tension. The highest point on this flat portion of the curve is sometimes called the *upper yield point*, and the point D is called the *lower yield point*. At this point the curve turns upward.

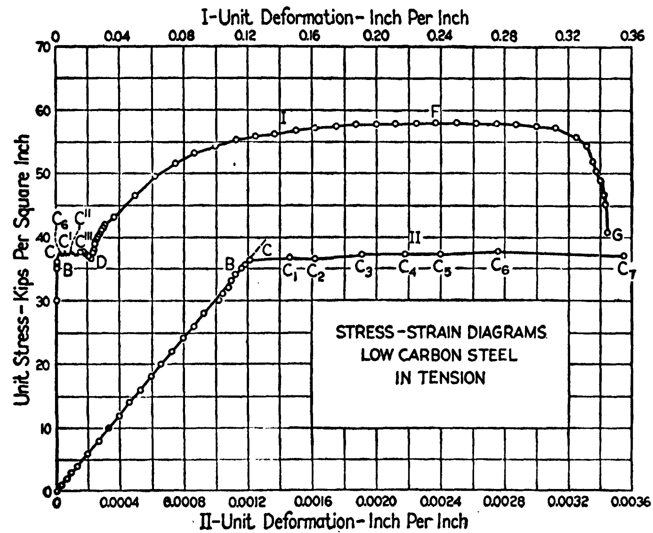


FIG. 40. Stress-strain diagrams for Table 7.

Yield point is determined by drop of the beam. With the testing machine running at a constant slow speed and the poise kept balanced automatically or by hand, a point is reached at which there is a sudden drop of the beam, and the poise must be run backward to balance. If the material is covered with scale, this scale begins to loosen and fall when any portion has reached the yield point.

Problems

29-1. Calculate the modulus of elasticity of the steel of Fig. 40 from the slope of Curve II.

29-2. Calculate the modulus of elasticity of the steel of Fig. 40 from the data of Table 7, using the stress interval from 4,000 to 34,000 psi.

Ans. 29,940,000 psi.

29-3. From the data of Table 7, find the total work in the 8-in. gage length when the unit stress changed from 4,000 to 34,000 psi. Use total load and total deformation. Divide by the volume to get the work per cubic inch. Check work per unit volume by Eq. (11.1).

Figure 41 shows stress-strain diagrams for three steel bars of quite different carbon content. The bar of Fig. 41,I (Appendix A) has 0.20% carbon. The curve is much like that of Fig. 40. The bar of Fig. 41,II (Appendix B) has 0.44% carbon. The yield-point stress is much higher than that of Fig. 41,I; the plastic deformation is less; and the maximum unit elongation is smaller. Figure 41,III is for steel of nearly 1% carbon. There is *no true yield point.* The elastic limit is at about 75,000 pounds per square inch. The maximum elongation is

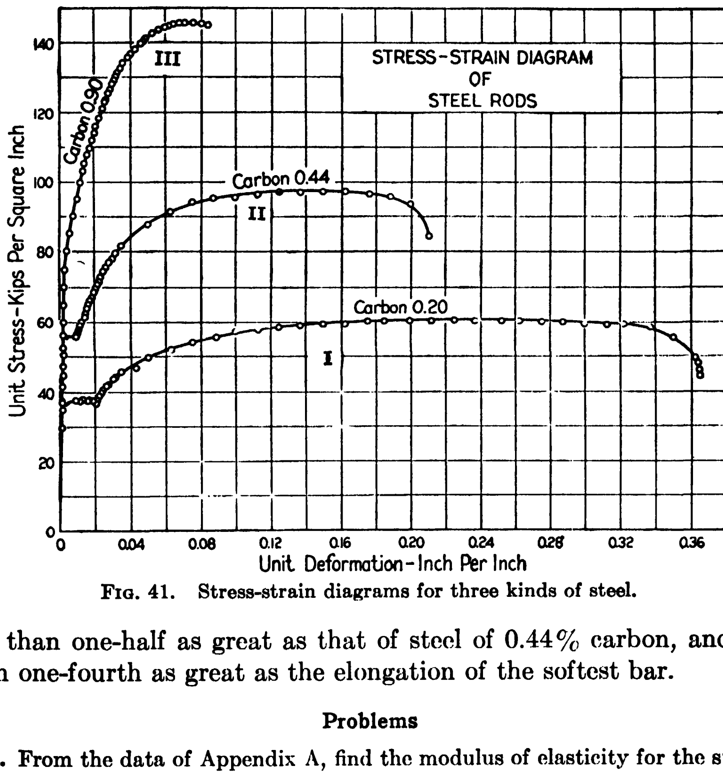


FIG. 41. Stress-strain diagrams for three kinds of steel.

less than one-half as great as that of steel of 0.44% carbon, and less than one-fourth as great as the elongation of the softest bar.

Problems

29-4. From the data of Appendix A, find the modulus of elasticity for the steel of Fig. 41,I, using the stress interval from 5,000 to 30,000 psi.

Ans. $E = 29{,}240{,}000$ psi.

29-5. From the data of Appendix B, find the modulus of elasticity for the 0.44% carbon steel of Fig. 41,II, using the stress interval from 1,000 to 31,000 psi.

29-6. From the data of Appendix C, find the modulus of elasticity for the SAE 1095 steel of Fig. 41,III, using a stress interval from 2,000 to 42,000 psi.

For many materials, such as steel or wrought iron, the percentage of elongation is an important factor. The bar of Fig. 40 (Table 7) was elongated 2.76 inches in a gage length of 8 inches. The percentage of

elongation was 34.5. The greatest relative elongation is in the portion of the bar which contains the neck. To show this, it is often advisable to subdivide the gage length into 1-inch intervals by punch marks and note that the inch containing the neck may elongate 60 per cent while the other seven inches in Fig. 42 will average perhaps 20 per cent. For this reason the ASTM has adopted standard gage lengths such as 8 inches or 2 inches.

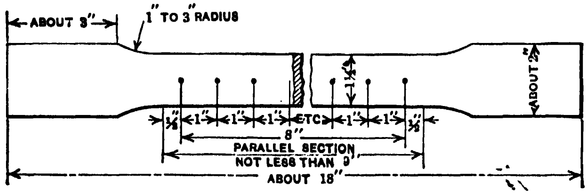


FIG. 42. Tension test bar—8-inch gage length.

To get the percentage of elongation, it is customary to fit the two pieces together after rupture and measure the total elongation of the gage length. For hard material with high ultimate strength, which breaks with inappreciable necking, this method gives an incorrect value for the percentage of elongation. The upper 4 inches of the steel bar of Appendix C was stretched 0.34 inch under unit stress of 145,040 pounds per square inch. The percentage of elongation before fracture was 8.5. After fracture, which took place in the lower 4 inches, the elongation of the upper 4-inch length had been reduced to 0.26 inch,

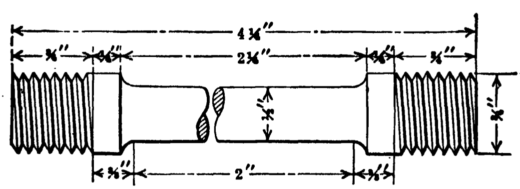


FIG. 43. Tension test bar—2-inch gage length.

or 6.5 per cent. A bar of this steel, when broken suddenly at almost the ultimate stress, shortens 2 per cent on the rebound. A bar of the same material shortened less than 1 per cent when the load was released gradually by reversing the machine. For softer steel the unit stress is materially reduced before fracture as the test bar necks. The rebound is relatively small and makes no appreciable error in the percentage of elongation. For a cold-rolled bar the rebound is about ½ of 1 per cent in the portion which is not fractured.

The percentage of reduction of area at the neck is an important constant of steel specifications. The final diameter of the neck of the soft

steel of Table 7 was 0.339 inch. The corresponding area is 0.0903 square inch.

$$\text{Reduction of area} = \frac{0.2820 - 0.0903}{0.2820} = 0.68 = 68 \text{ per cent}$$

Problems

29-7. Find the percentage of reduction of area and of elongation for the bar of Appendix A. Use the square of the diameter instead of the area.
Ans. 66.5 per cent.

29-8. Find the reduction of area for the bar of Appendix B and the bar of Appendix C. *Ans.* 44.5 per cent; 11.6 per cent.

The point F of Fig. 40,I at unit stress of 58,030 pounds per square inch indicates the ultimate strength of this test bar of low-carbon steel. The point G at the end of the curve is the breaking strength or stress at rupture. For brittle materials, breaking strength and ultimate strength practically coincide. For ductile materials, the ultimate strength is much higher than the breaking strength. Used in connection with stress, the word *ultimate* means the *greatest stress,* not the last stress.

When a bar is tested in tension, the cross section decreases uniformly throughout the entire gage length until the ultimate strength is reached. Finally some section begins to decrease much faster than the remainder of the gage length. This reduced section is called the *neck.*

On account of the smaller section's having to carry the load after the bar begins to neck, the total load is reduced. After the diameter at the neck has become somewhat smaller than that of the remainder of the gage length, the total stress becomes so low that there is no additional elongation except in the portion at the neck. For the last part of the test, the uniform portions of the bar shorten slightly under the reduced load.

Problem

29-9. What are the ultimate strength and the breaking strength of each bar of Fig. 41? Read the values from the curves.

The unit stresses of the preceding tables and curves were calculated by dividing the total load by the area of the cross section at the beginning of the test. This is the usual method of calculation and unit stress always is so understood, unless otherwise designated. When it is desirable to distinguish from unit stress calculated in some other way, the unit stress obtained from the original area may be called the *nominal unit stress.*

On account of the permanent reduction of area of a ductile material after passing the yield point, the *actual unit stress*, which is calculated by dividing the total load by the area of the cross section as loaded, may be much larger than the nominal unit stress from the original area.

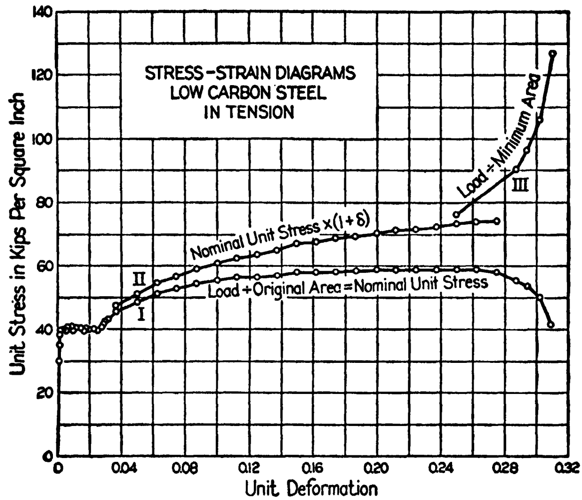


FIG. 44. Nominal and actual unit stress.

Problems

29-10. From Table 7, calculate the actual unit stress when the elongation was 2.55 in. *Ans.* 88,200 psi.

29-11. From Table 7, find the actual breaking stress, using information on the last line of the table.

Before necking begins, the actual unit stress may be calculated from the nominal unit stress and the unit deformation. After reaching the yield point, the volume remains practically constant. If A represents the original area of cross section, the volume of a portion 1 inch in length is equal to A cubic inches. If A' is the area of the cross section when the original inch of length has been stretched to a length of $1 + \epsilon$, the volume is $A'(1 + \epsilon)$.

$$A = A'(1 + \epsilon) \qquad A' = \frac{A}{1 + \epsilon}$$

$$\text{Actual unit stress} = \frac{P}{A'} = \frac{P}{A}(1 + \epsilon)$$

$$\text{Actual unit stress} = \text{nominal unit stress} \times (1 + \epsilon)$$

Yielding of steel takes place by shearing deformations along microscopic planes. These planes of shear are called *slip planes*. The direc-

tion of the slip planes in a small portion of a steel bar varies greatly since the deformation depends upon the orientation of the steel crystals. A group of slip planes, however, follows the direction of maximum shearing stress. If the steel is covered with mill scale, the deformation caused by a group of slip planes fractures the scale and appears as a line on the surface. As the loading is continued and the area of the surface which has reached the yield point enlarges, a con-

TABLE 8. COMPRESSION TEST OF CAST IRON

Bar 0.792 inch square; area, 0.627 square inch. Gage length, 8 inches. Bar previously loaded to 51,000 pounds per square inch. Released to 100 pounds per square inch and held at this stress for 10 minutes before beginning test.

Total load, pounds	Unit stress, pounds per square inch	Extensometer reading	Deformation		
			Divisions	Inches	Unit
63	100	3,294	0	0	0
1,254	2,000	3,260	34	0.00068	0.000085
2,508	4,000	3,204	90	180	225
3,762	6,000	3,142	152	304	380
5,016	8,000	3,092	202	404	505
6,270	10,000	3,035	259	0.00518	0.000647
7,524	12,000	2,981	313	626	782
8,778	14,000	2,922	372	744	930
10,032	16,000	2,866	428	856	0.001070
11,286	18,000	2,806	488	976	1220
12,540	20,000	2,759	535	0.01070	0.001337
13,794	22,000	2,700	594	1188	1485
15,048	24,000	2,635	659	1318	1648
17,556	28,000	2,522	772	1544	1930
18,810	30,000	2,460	834	1668	2085
20,064	32,000	2,408	886	0.01772	0.002215
21,318	34,000	2,350	944	1888	2360
22,572	36,000	2,297	997	1994	2492
23,826	38,000	2,236	1,058	2116	2645
25,080	40,000	2,180	1,114	2228	2785
26,334	42,000	2,110	1,184	0.02368	0.002960
27,588	44,000	2,050	1,244	2488	3110
28,842	46,000	1,985	1,309	2618	3272
30,096	48,000	1,918	1,376	2752	3440
31,450	50,160	1,820	1,474	2948	3685
31,977	51,000	1,780	1,514	0.03028	0.003785

siderable portion of the scale may break from the bar.

Figure 45 shows the yielding of two 2½- by ¼-inch bars of soft steel in tension. Before testing, all the plain bar and all the notched bar except about 2 inches at the middle had been painted with a thin coat of whiting mixed with water. When tested, a fine line first appears and extends gradually across the bar. As the stretching is continued, other lines form; wide bands of scale bend up and finally break off. This accounts for the relatively large areas which appear entirely dark.

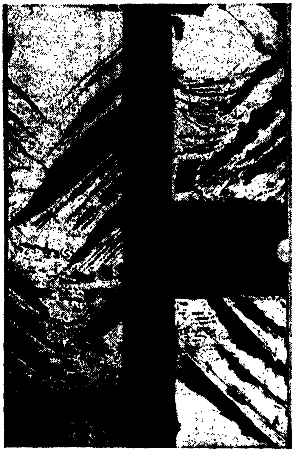


FIG. 45. Plain bar and notched bar in tension.

The capacity of a material to absorb energy is of importance when dealing with impact loads. The area under the stress-strain diagram is a measure of this capacity and is frequently used as a guide to the toughness of a material. A comparision of the three steels of Fig. 41 can be obtained by a mental estimate of the areas.

30. Stress-Strain Diagrams of Other Materials. Table 8 gives the results of a test of a cast-iron bar which was about ¾ inch square and 10 inches long. The bar was tested as cast without removing the hard skin at the surface. The ends were cut off square and the bar was tested on parallel compression plates without ball-and-socket joints.

These cast-iron test pieces have internal stresses. When loaded the first time, the stress diagram is irregular and begins to curve at very low loads. After one loading the diagram approaches a straight line under loads which greatly exceed the allowable stress. The stress-strain diagram up to 40,000 pounds per square inch is plotted on Fig. 46.

Table 9 gives the average of six tests of cast iron in tension which were made at the Watertown Arsenal. While the diagram for any single test piece would show considerable deviation from a smooth curve, the average of six shows very little deviation.

Problems

30-1. From Table 9, find the modulus of elasticity for cast iron in tension by means of the interval 1,000 to 6,000 unit stress. *Ans.* $E = 15{,}700{,}000$ psi.

30-2. From Table 8, find E for cast iron in compression for the intervals 2,000 to 12,000 psi and 4,000 to 14,000 psi.

Ans. $E = 14{,}350{,}000$ psi; $E = 14{,}180{,}000$ psi.

Table 10 gives the records of a test of 28-day concrete with a water-cement ratio of 1.3 by volume. Three consecutive runs were made.

Each of the first two carried the unit stress to 1,272 pounds per square inch, while the last run was continued to failure at 3,015 pounds per square inch. The concrete curve of Fig. 46 was made from the last run.

When steel is stressed beyond the yield point and when timber is stressed considerably beyond the elastic limit, deformation continues to increase with no increase of load. This is called *creep* or *flow*. For concrete, creep occurs at very low stresses. If the machine is stopped to take extension readings, the form of the curve depends largely upon the elapsed time. Different observers may get quite different curves.

TABLE 9. TENSION TEST OF CAST IRON

From Watertown Arsenal, "Tests of Metals," 1905. Average of six tests, specimens 8014, 8041, 8050, 8051, 8053, and 8063. Diameter, 1.129 inch; area, 1 square inch. Gage length, 10 inches. Bars machined to size, removing all the hard skin.

Load, pounds per square inch	Elongation, inches	
	In gage length	Per inch length
1,000	0	0
2,000	0.00056	0.000056
3,000	0.00115	0.000115
4,000	180	180
5,000	245	245
6,000	0.00318	0.000318
7,000	390	390
8,000	470	470
9,000	550	550
10,000	635	635
11,000	0.00723	0.000723
12,000	815	815
13,000	912	912
14,000	0.01005	0.001005
15,000	1118	1118
16,000	0.01225	0.001225
17,000	1348	1348
18,000	1488	1488
19,000	1633	1633
20,000	1795	1795
21,000	0.01947	0.001947
22,000	2126	2126
23,000	2364	2364
24,000	2570	2570
26,450 Average ultimate load		

On this test the crosshead of the testing machine was run at 0.05 inch per minute and the two strain gages read simultaneously.

Problems

30-3. From the third run of Table 10, find E for 28-day concrete for the intervals 6,000 to 16,000 lb, 8,000 to 18,000 lb, and 10,000 to 20,000 lb total load.
Ans. E = 3,400,000 psi; 3,270,000 psi; 3,100,000 psi.

30-4. Solve Prob. 30-3 from the first run and from the second run.

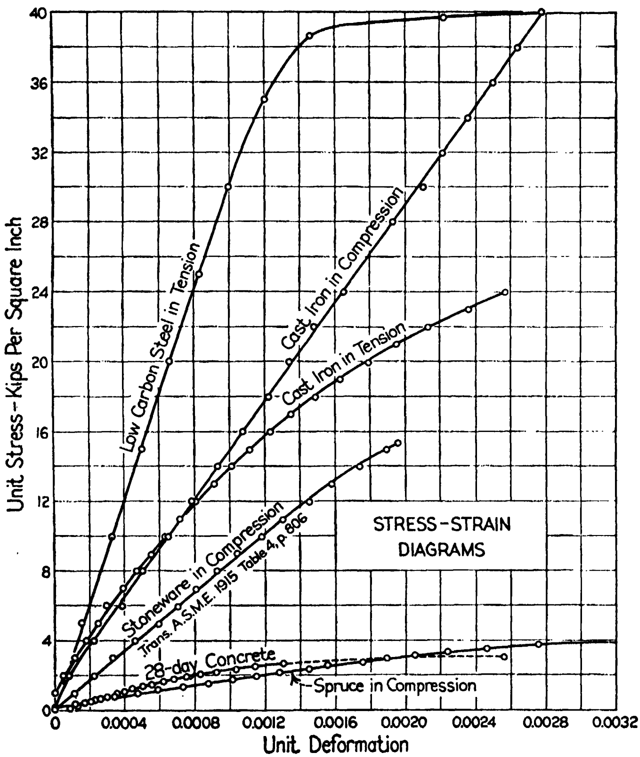


FIG. 46. Stress-strain diagrams for several materials.

Since it is somewhat difficult to determine the proportional elastic limit accurately, especially in hard steel where there is a wide range between this stress and the yield point and in materials (such as cast iron) which have no yield point, the late Prof. J. B. Johnson proposed another point which he called the *apparent elastic limit.* He defined the apparent elastic limit as "the point on the stress diagram at which the rate of deformation is 50 per cent greater than at the origin." It is that point on the curve at which the slope of the tangent from the *vertical* is 50 per cent greater than that of the straight-line part of the

TABLE 10. COMPRESSION TEST OF CONCRETE CYLINDERS

Diameter, 6 inches; length, 12 inches. Mix 1:4¼:4 by volume, water cement 1.3 by volume. Compression measured by two Berry strain gages. Crosshead speed, 0.05 inch per minute. Continuous run, gages read without stopping machine. Cylinders made and tested by Profs. J. R. Shank and G. E. Large.

Total load, pounds	Unit stress, pounds per square inch	Unit deformation		
		First run	Second run	Third run
2,000	71	0	0.000079	0.000088
4,000	141	0.000015	88	94
6,000	212	32	0.000108	0.000109
8,000	283	53	120	128
10,000	354	73	139	146
12,000	424	0.000095	0.000160	0.000171
14,000	495	0.000117	179	191
16,000	565	141	202	213
18,000	636	167	226	236
20,000	707	191	244	260
22,000	777	0.000212	0.000267	0.000280
24,000	848	235	286	304
26,000	919	259	312	326
28,000	989	284	335	349
30,000	1,060	310	358	373
32,000	1,131	0.000340	0.000381	0.000403
34,000	1,201	367	403	419
36,000	1,272	398	430	445
38,000	1,342	....	....	470
40,000	1,413	....	....	507
44,000	1,555	....	....	0.000557
48,000	1,696	....	....	619
52,000	1,837	....	....	683
56,000	1,980	....	....	753
60,000	2,120	....	....	835
64,000	2,260	....	....	0.000924
68,000	2,400	....	....	0.001035
72,000	2,544	....	....	1150
76,000	2,685	....	....	1300
80,000	2,825			
84,000	2,965			
85,400	3,015 Failed	....	....	0.002560

curve. In some investigations of the strength of materials, the apparent elastic limit has been found useful in comparing the results of different tests.[1]

Figure 47, which is drawn from the concrete test of Table 10, illustrates the graphical determination of Johnson's apparent elastic limit.

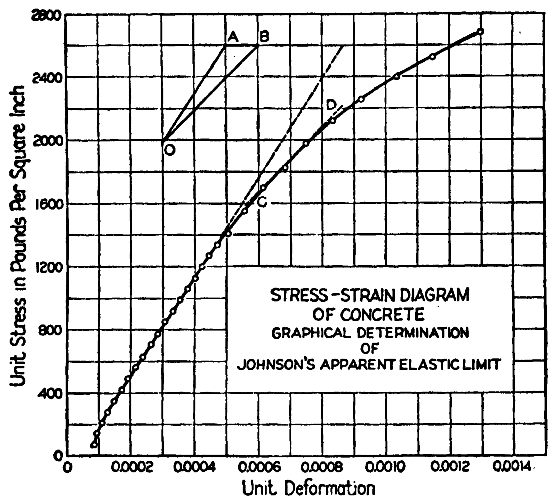


FIG. 47. Graphic determination of Johnson's elastic limit from Table 10.

Problems

30-5. Find Johnson's apparent elastic limit for the hard steel of Appendix C.
Ans. 70,000 psi.

30-6. Find Johnson's apparent elastic limit for the longleaf yellow pine of Table 4.

In Fig. 48 is given the stress-strain diagram for aluminum alloy 24S-T at three different temperatures, 77, -109, and -313°F. Each curve is plotted from observations on at least two (and sometimes seven) tests,[2] using different types of strain gages to check. The three diagonal straight lines starting at 0.002 deformation are drawn parallel to the straight-line portion of the stress-strain diagrams. The stress obtained at the intersection with the diagrams is called the *yield strength*. By definition, yield strength is the stress which produces a permanent set of 0.2 per cent of the initial gage length.[3] It should be

[1] See the work of H. F. Moore in *Bulletin* 42 and Albert J. Becker in *Bulletin* 85 of the University of Illinois Engineering Experiment Station.

[2] These were obtained through the courtesy of John L. Zambrow, research engineer at the Ohio State University Engineering Experiment Station.

[3] See ASTM, Standard Methods of Tension Testing, ASTM, E8-42.

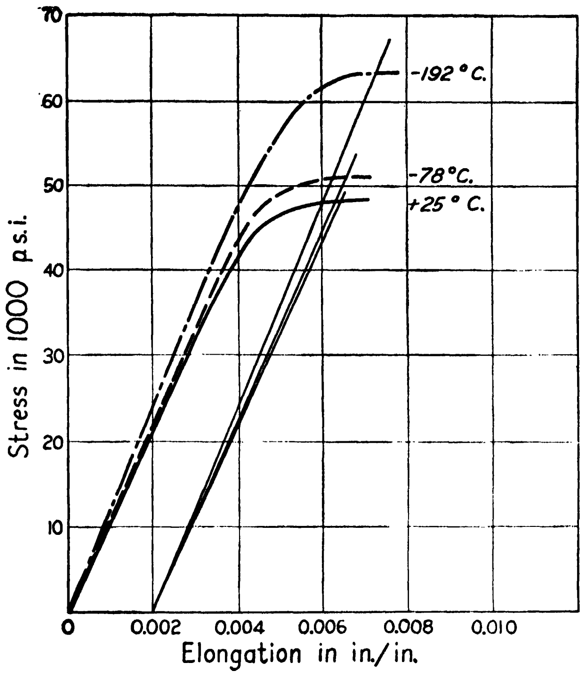


FIG. 48. Stress-strain diagrams for aluminum alloy 24S-T.

noted that yield strength is not yield point, nor is it elastic limit. Additional results of these tests for 24S-T are[1]

Temperature, °C	E, psi	Ultimate strength, psi	Yield strength, psi	Elongation in 2 in., per cent	Reduction of area, per cent
+25	10,870,000	69,875	48,500	19.8	30.3
−78	11,110,000	72,312	51,063	20.8	25.5
−192	11,970,000	87,312	63,125	21.3	20.5

31. Other Factors Affecting Stresses. There are many variables which may affect the stress of a specimen or of a structure. Only a few can be mentioned here.

In calculating unit stress it has been assumed that the stress is uniform over the entire cross section. This is true in a rod of uniform cross section at some distance from the surface at which the external

[1] The Alcoa Structural Handbook, published by the Aluminum Company of America, gives as the composition of 24S-T copper 4.5 %, manganese 0.6 %, magnesium 1.5 %, and aluminum the remainder.

force is applied, provided the resultant force acts along the axis of the body. In order to have the stress uniformly distributed and the same at all sections, test bars are made of uniform section throughout a portion somewhat greater than the gage length. The bar tapers gradually from the larger to the smaller sections.

The broken lines in Figs. 49 and 50 represent the flow of stress in the bar. At sections C the lines are crowded, indicating a greater intensity

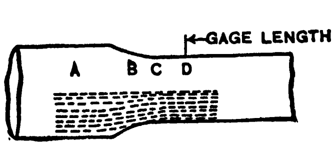


FIG. 49. Stress distribution in test bar.

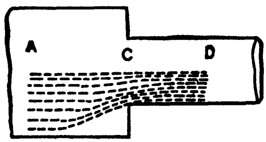


FIG. 50. Abrupt change of section.

of stress near the surface. If the gage length extended to C at the end of the parallel portion, the measured elongation would be too great, since the stress in the outer surface (to which the gages are applied) is greater than the average of the section. The more abrupt the change in section, the greater the inequality of stress.

These abrupt changes is sections cause "stress concentrations," a phenomenon very disconcerting to the designer.

Figure 51 represents a plate of uniform width and thickness with a circular hole at the middle of its width. The plate is subjected to tension producing uniform tensile stress over any section located some distance from the hole. The arrows at the top represent this uniform stress at one section. The arrows which begin at the horizontal line through the center of the hole represent the stress at that section. The problem of the stress distribution of a plate of this kind has been solved, and the solution has been verified by photoelastic experiments.[1] For a plate of considerable width in comparison with the diameter of the hole, the maximum stress at the tangent is three times the stress in the uniform portion. This stress is shown in Fig. 51 by the two longest arrows. At one-fifth the radius from the tangent, the stress is 2.070 times the stress in the uniform portions. The stress at a distance from the tangent equal to the radius is 1.219 times the stress in the uniform

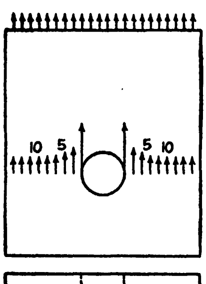


FIG. 51. Variation of stress near a circular hole.

[1] See TIMOSHENKO, S., "Strength of Materials," D. Van Nostrand Company, New York, 1941, Part II, Chap. VII; SEELY, "Advanced Mechanics of Materials," John Wiley & Sons, Inc., New York, 1932, Chap. XI.

portions. The stress at a distance from the tangent equal to the diameter is 1.074 times that of the uniform portions.

The ultimate strength of a body at a point where the section changes depends upon the ductility of the material. A rod of cast iron or other *nonductile* material of the form of Fig. 50 will fail at section C, where there is a concentration of stress near the surface. The more abrupt the change of section, the greater the concentration and the easier the failure.

On the other hand, ductile material subjected to tension is not likely to fail at a section at which there is an abrupt change. While the stress at the hole in Fig. 51 is three times the average, this overstressed

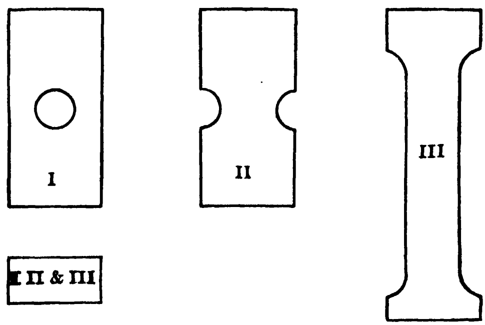


FIG. 52. Reduced sections.

material soon reaches the yield point. The ductile metal receives a permanent set and the stress increases very little with any further deformation. When the load is removed, this material is under compression and produces some tension in the remainder of the section. A ductile substance necks before it fails. The larger portions of Figs. 49 and 50 prevent necking in the smaller portions to the right of C. Rods of ductile material with short reduced area, such as I and II of Fig. 52, may show a higher ultimate strength than a rod for which the minimum section is longer, as in III (Fig. 52). In rods I and II the portion with the minimum section is close to the larger sections which hinder the necking. In rod III, on the other hand, most of the portion of minimum section is so far removed from the larger sections that necking takes place without hindrance.

Figure 53 shows part of the stress-strain diagram of low-carbon steel (carbon, 0.20%). The test piece was taken from the same rod as that of the lower curve of Fig. 41. The plastic stage ended at unit elongation of 0.0199 and unit stress of 38,059 pounds per square inch (Table 11). The loading was continued to a unit elongation of a little over

2 per cent and unit stress of nearly 40,000 pounds per square inch. The load was then released to 1,000 pounds per square inch. The specimen was immediately reloaded (*C* to *F* and *D*) and then unloaded to 100 pounds per square inch, where it was allowed to remain overnight. The portion *EFH* indicates a new higher yield point. The stress was then reduced to 100 pounds per square inch at *I*.

After standing 48 hours under a small load, other loads were applied which are not given in Fig. 53 or in Table 11. One loading was carried up to 43,000 pounds per square inch, which is only a little above *D*, the maximum of the second run. When this load was relieved, there was no additional set. The next loading was carried to 46,800 pounds

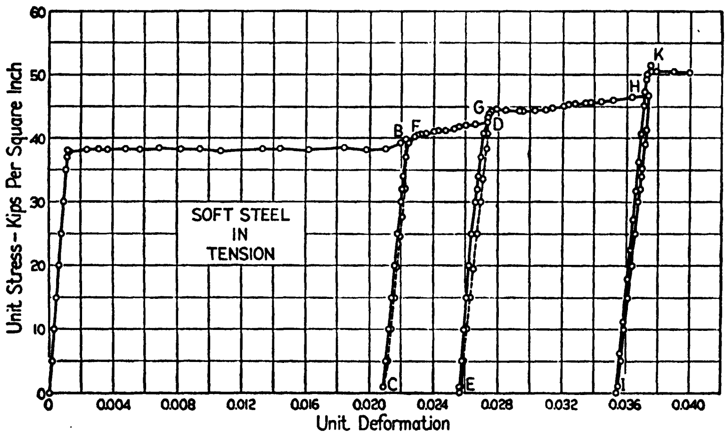


FIG. 53. Repeated loading of soft steel from Table 11.

per square inch, which was nearly to *H*, the maximum of the third loading. After this load was released, there was an additional permanent set of 0.0001 inch in the gage length. The last loading, represented by curve *IHK*, was then applied. The curve continues nearly straight up to 50,000 pounds per square inch. There is a marked upper yield point at *K* at 51,530 pounds per square inch.

When a ductile material at ordinary temperatures is permanently deformed by the application of force, it is said to be *cold-worked*. Cold-working raises the elastic limit and yield point of low-carbon steel. It also raises the ultimate strength in terms of the area after cold working. Cold-worked material is said to be *work hardened*. One great advantage in the use of low-carbon steel depends upon this property. When an inaccurate construction or unusual loading throws excessive stress on some part of a structure made of soft steel, some permanent deformation may occur but the structure is not weakened in the least.

TABLE 11. REPEATED TENSION IN SOFT STEEL

Test piece from rod used for Appendix A, lower stress-strain diagram of Fig. 34. Crosshead speed, 1/40 inch per minute.

Unit stress, lb. per sq. in.	Unit deformation in 0.000001 in.	Unit stress, lb. per sq. in.	Unit deformation 0.000001 in.	Unit stress, lb. per sq. in.	Unit deformation 0.000001 in.
3 P.M., June 28		3:59 P.M.		Reversed	
100	0	40,180	22,815	41,422	37,307
1,000	25	41,084	24,042	38,962	37,217
5,000	145	42,032	26,062	35,440	37,072
10,000	320	42,664	27,355	30,000	36,812
15,000	482	4:21 P.M., June 28		25,000	36,587
		Reversed			
20,000	650	38,443	27,295	20,000	36,360
25,000	820	33,409	27,120	15,000	36,120
30,000	970	29,955	26,977	10,000	35,900
35,000	1,130	25,000	26,772	5,000	35,645
36,610	1,185	19,548	26,555	1,000	35,417
				100	35,367
38,262	1,235	15,000	26,317	9:58 A.M., June 29	
37,923	1,325	10,000	26,069	3 P.M., July 1	
38,126	2,430	5,000	25,837	6,772	35,592
38,104	3,712	1,000	25,627	11,287	35,765
38,059	19,860	100	25,570	18,059	36,007
		4:41 P.M., June 28		22,573	36,192
39,278	21,980	8:50 A.M., June 29		27,088	36,357
39,847	22,340	1,000	25,582		
Reversed		5,000	25,707	31,603	36,547
32,054	22,210	10,000	25,895	36,343	36,727
27,765	22,072	15,000	26,067	40,632	36,925
24,605	21,940	20,000	26,265	45,140	37,115
				47,404	37,215
19,814	21,755	25,000	26,462		
15,000	21,535	30,000	26,672	49,278	37,292
10,000	21,330	32,000	26,770	50,113	37,330
5,000	21,105	34,000	26,852	50,790	37,422
1,000	20,885	37,043	26,987	51,530	37,505
3:59 P.M.				50,677	37,530
5,000	21,010	40,750	27,162		
10,000	21,200	43,386	27,322	50,609	37,642
15,000	21,380	44,130	27,477	50,519	37,972
20,000	21,567	44,492	27,640	50,384	38,942
25,000	21,785	44,458	28,492	50,452	39,917
30,000	22,002	44,323	29,122		
32,000	22,080	45,011	32,092		
34,000	22,180	46,117	35,212		
37,170	22,312	46,456	36,312		
39,232	22,460	46,840	37,407		

32. Ultimate Strength in Shear. Figure 54 shows one type of apparatus for determining the ultimate shearing strength of metal rods. The test rod A fits closely in two hollow cylinders B and B', which are made of hardened steel. The hard-steel shear plate C is placed across the test piece between adjacent ends of the hollow cylinders, with its semicircular opening fitting around the upper half of the specimen.

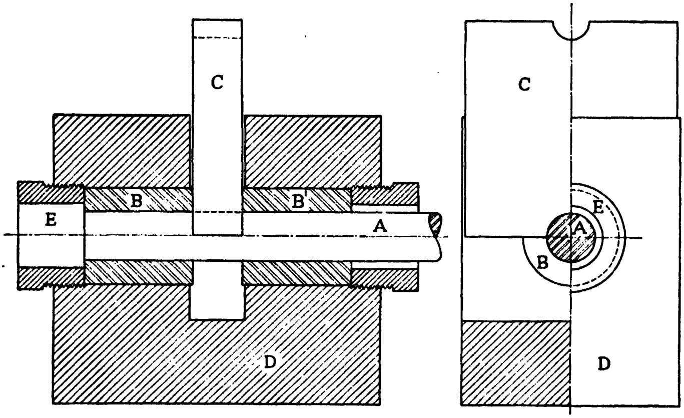


FIG. 54. Apparatus for testing shearing of metal rods.

The hollow cylinders are mounted in a heavy, rigid block of cast iron or steel D. A rectangular slot, a little wider than the thickness of the shear plate, is cut across the block. A cylindrical hole, at right angles to the slot, is bored lengthwise through the block. The inner portion of each half of the hole for a length a little less than the length of a hollow cylinder B is finished to fit closely to the hollow cylinder. The outer ends of the hole are threaded. A hollow threaded cylinder E, in each end, adjusts the smooth hollow cylinder B (or B') and holds it firmly against pressure outward.

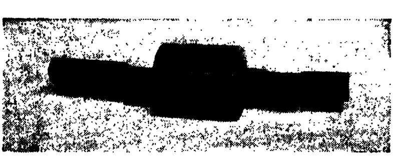


FIG. 55. Cylinder of shear apparatus and tested steel rod.

The apparatus is placed on the weighing table of a universal testing machine and the load is applied by the movable crosshead to the shear plate C. Before the load is applied, the threaded cylinders are turned to bring the ends of the smooth cylinders B and B' against the shear plate. The adjustment is made to bring the plate near the middle of the slot, so that it will touch only the test rod and the smooth ends of the shear cylinder B and B'.

For double shear the rod extends entirely through each hollow cylinder, to prevent bending as much as possible. For single shear the rod

extends entirely through one hollow cylinder but does not quite reach the other hollow cylinder.

Figure 55 shows one of the hollow cylinders with a ¾-inch soft-steel rod which has been loaded to its ultimate strength at two places. With the deformation shown, the rod is still able to support almost its ultimate load. For a similar rod tested, the maximum load was 37,700 pounds. The machine was stopped after the load had dropped to 37,500 pounds. A straight line which had been drawn longitudinally on the side of the rod before loading was found to be displaced downward on the slug a distance of 0.11 inch below its position on the rod.

Problems

32-1. What was the ultimate shearing strength of the rod mentioned above? *Ans.* Ultimate s_s = 42,670 psi.

32-2. The rod of Prob. 32-1 tested in single shear gave for the maximum reading 19,100 lb. Find the ultimate shearing strength. *Ans.* s_s = 43,230 psi.

32-3. With the average results from Probs. 32-1 and 32-2, what would be the allowable unit shearing stress with a factor of safety of 5?

Figure 56 shows the apparatus for measuring shearing strength of timber parallel to the grain, which was developed by the plate *C*. The second shearing surface is on the hardened-steel plate *B*. The shearing

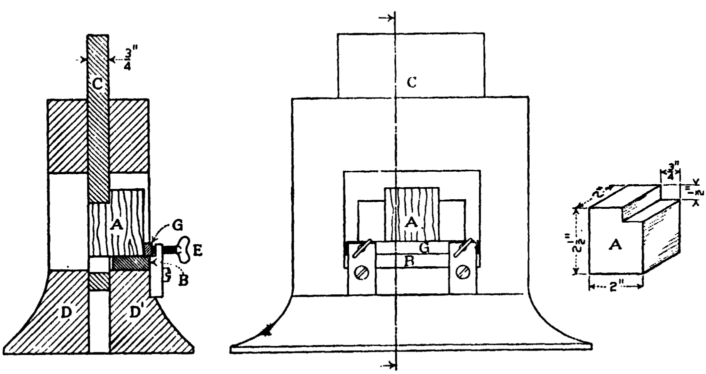


FIG. 56. Shear apparatus for timber.

surface on *B* is set back ⅛ inch from the shearing surface of *C*, instead of being in direct contact with it as in the apparatus for shearing steel. If the grain of the block is not exactly perpendicular to the plane of the base, some fibers might begin on *B* and end under *C*. These fibers would be tested in compression instead of shear, and the measured results would be too high. With a clearance of ⅛ inch and reasonable care in the preparation of the specimen, this error will not occur.

Figure 56 also shows the form of test piece specified by the ASTM[1]. A notch of ¾ inch gives room for the entire thickness of the test plate to act in bearing. The rectangular bar *B*, which is adjusted by the screws *E*, holds the lower end of the test block in the desired position. If the test block is held vertically until a small load is applied, the friction of the shear plate will continue to hold it.

Problems

32-4. A test piece of longleaf yellow pine (Fig. 56) was 1.89 in. wide and 1.92 in. high up to the notch. It failed in shear under a load of 5,300 lb. Find the ultimate shearing strength of this piece. *Ans.* $s_s = 1{,}460$ psi.

32-5. A second test piece cut from the same stick as that of Prob. 32-4 was 1.89 in. wide and 1.90 in. high to the notch. The ultimate load in shear was 5,150 lb. Find the shearing strength parallel to the grain. *Ans.* $s_s = 1{,}435$ psi.

32-6. A rectangular parallelepiped from the same stick as Probs. 32-4 and 32-5 was 1.88 in. long and 1.62 in. wide. (The direction of shear was at right angles to that of preceding test pieces.) The ultimate load was 5,200 lb. Find the unit shearing strength of this sample. *Ans.* $s_s = 1{,}707$ psi.

32-7. A rectangular parallelepiped of white pine, 2.18 in. long and 2.05 in. wide, failed by shear under a load of 4,350 lb. Find the shearing strength. *Ans.* $s_s = 973$ psi.

32-8. A second block of the same white-pine piece was 2.18 in. long and 1.81 in. wide. It failed under a load of 4,250 lb. Find the ultimate shearing strength of this white pine. *Ans.* $s_s = 1{,}077$ psi.

33. Factor of Safety. In Art. 6, the allowable unit stress was said to be based on the judgment of some competent authority. These judgments depend upon tests of the materials and upon experience in actual use.

Working stresses should never exceed the elastic limit and should be only a small fraction of the ultimate strength. The ratio of the ultimate strength of a material to the allowable working stress is called the *factor of safety*. If the ultimate tensile strength of a given grade of steel is 64,000 pounds per square inch and the elastic limit is 32,000 pounds per square inch, the factor of safety based on the ultimate strength is 4, and that based on the elastic limit is 2.

The value of the factor of safety depends upon a great number of conditions. Some of these are

1. Repeated stresses slightly beyond the elastic limit will finally cause failure; therefore a body subjected to a variable load should have its allowable stresses well below that limit. The greater the variation of stress, the smaller should be the allowable unit stress.

2. Dynamic stresses may be encountered. The stress due to impact

[1] *ASTM Standards*, 1946, D143-27.

is sometimes "calculated" by multiplying the "dead load" by a "factor," without analyzing the forces or vibrations sufficiently.

3. Localized stresses caused by abrupt changes in the shape or cross section of a member may be significant and require consideration.

4. Materials used at higher or lower temperatures may have entirely different properties than when used in the familiar habitat of the Temperate Zone.

5. The factor of safety must be sufficiently large to allow for any deterioration of the material during the time which it is to be used. This includes the decay of timer, the corrosion of metal, injury from frost, and electrolysis.

6. The uniformity of the material must be taken into account in deciding what factor of safety to use. If test pieces from a batch of structural steel manufactured under well-controlled conditions give an ultimate strength of 57,000 pounds per square inch, none of the steel of this batch may vary more than a few hundred pounds from this figure. On the other hand, the variation of timber sufficiently good to pass a reasonable inspection may be as much as 50 per cent of the average ultimate strength. An engineer, in designing a structure to be built under competent supervision, may use considerably higher unit stresses than he would risk when such supervision is lacking.

7. The factor of safety must depend also upon the damage which would occur if the material should fail. A workman might use a

TABLE 12. TYPICAL MECHANICAL PROPERTIES OF ALUMINUM ALLOYS
(SOURCE: Alcoa Structural Handbook)

Alloy and temper	Tension, psi		Elongation in 2 in., per cent	Compression, psi	Shear, psi		Fatigue endurance limit, psi[1]
	Yield strength	Ultimate strength		Yield strength	Yield strength	Ultimate strength	
Wrought							
3SO	6,000	16,000	40	6,000	4,000	11,000	7,000
3S½H	18,000	21,000	16	18,000	12,000	14,000	9,000
17S-T	37,000	60,000	22	37,000	22,000	36,000	18,000
24S-T	46,000	68,000	22	46,000	28,000	41,000	18,000
61S-T	39,000	45,000	15	39,000	26,000	30,000	13,500
Cast							
43	9,000	19,000	6	10,000	......	14,000	6,500
214	12,000	25,000	9	12,000	......	20,000	5,500

[1] Endurance limits are based on 500,000,000 cycles of completely reversed stress on the R. R. Moore machine.

plank with a small factor in a scaffold 3 feet above ground but would demand an ample factor if failure meant a fall of 100 feet.

8. The factor of safety must allow some margin for unexpected and unreasonable loads. That part of the factor of safety which makes allowance for lack of ordinary judgment on the part of persons using the machine or structure is called the *fool factor.*

34. Some Specifications. In Table 12 a few selected aluminum alloys are listed with some typical properties. Wrought alloys produced by rolling, extruding, drawing, or forging are indicated by S following the number which identifies the particular chemical composition. A fully annealed material is indicated by 0, and a fully hardened one by H. Intermediate grades of hardness are denoted by fractions. When an alloy has been fully heat-treated and aged it is indicated by T.

Table 13 gives a few specifications of materials as adopted by the ASTM, 1946. The values listed are minimum except in the case of ultimate tensile strength, where both minimum and maximum are given.

Example

A ¾-in. diameter nickel-steel bar when tested in tension carried a maximum load of 35,200 lb and a yield-point load (by drop of beam) of 17,820 lb. An original 8-in. gage length elongated 1.44 in. and the final diameter of the neck was 0.640 in. Does this meet the specifications for grade B nickel-steel plates for boilers and other pressure vessels?

Original area = 0.442 sq in. Final area = 0.322 sq in.

Ultimate strength = 79,600 psi. OK. Yield point = 40,300 psi. Not satisfactory.

Elongation = 18 per cent. Not satisfactory, as this strength requires 20.7 per cent.

Reduction of area = 27.1 per cent. No requirement stated for this.

Problems

34-1. A test bar from structural nickel steel is 0.505 in. in diameter, reaches a yield point at 11,220 lb, and carries a maximum load of 19,200 lb. The 2-in. gage length elongates to 2.42 in. and the final diameter is 0.42 in. Does this meet specifications?

34-2. A steel bar is ⅝ in. in diameter and is tested in tension. The yield point is reached at 11,000 lb, the ultimate at 20,000 lb, and the breaking load at 15,000 lb. The final length of an 8-in. gage is 10.25 in. and the final diameter is 0.380 in. Does this meet the requirements for bridges and buildings?

34-3. Find the actual and nominal breaking stress in Prob. 34-2.

34-4. A medium-hard drawn copper wire 0.162 in. in diameter carries a maximum load of 1,060 lb before breaking. The 60-in. gage length elongates to 60.72 in. Will this meet specifications?

34-5. A standard 0.505-in. diameter bar of 24S-T aluminum has a total load at 0.002 in. per in. elongation of 9,360 lb, and carries a maximum load of 14,050 lb. The elongation in a 2-in. gage length is 0.46 in. How does this compare with the typical properties given in the table?

TABLE 13. SPECIFICATIONS FOR SOME METALS ADOPTED BY THE AMERICAN SOCIETY FOR TESTING MATERIALS, STANDARDS, 1946

Material		Tensile strength, psi	Yield point, psi (but in no case less than)	Per cent elongation		Per cent reduction of area
				In 8 in.	In 2 in.	
Steel for bridges and buildings (ASTM, A7-46)		60,000 to 72,000	0.5 tens. str. (33,000)	$\frac{1,500,000}{\text{tens. str.}}$	22	
Structural rivet steel, (ASTM, A141-39)		52,000 to 62,000	0.5 tens. str. (28,000)	$\frac{1,500,000}{\text{tens. str.}}$		
High-strength structural rivet steel (ASTM, A195-41)		68,000 to 82,000	0.5 tens. str. (38,000)	$\frac{1,600,000}{\text{tens. str.}}$ (20)		
Structural nickel steel (ASTM, A8-46)		90,000 to 115,000	0.5 tens. str. (55,000)	$\frac{1,600,000}{\text{tens. str.}}$		30
Nickel-steel plates for boiler and other pressure vessels (for firebox) (ASTM, A203-46)	Grade A	65,000 to 77,000	0.55 tens. str. (37,000)	$\frac{1,650,000}{\text{tens. str.}}$	$\frac{1,750,000}{\text{tens. str.}}$	
	B	70,000 to 82,000	0.55 tens. str. (40,000)	$\frac{1,650,000}{\text{tens. str.}}$	$\frac{1,750,000}{\text{tens. str.}}$	
	C	75,000 to 87,000	0.55 tens. str. (43,000)	$\frac{1,650,000}{\text{tens. str.}}$	$\frac{1,750,000}{\text{tens. str.}}$	
Medium-hard-drawn copper wire (ASTM B2-40)	Diameter, in.			In 10 in.	In 60 in.	
	0.4600	42,000 to 49,000		3.75		
	0.2294	48,000 to 55,000		2.25		
	0.1620	49,000 to 56,000			1.15	
	0.05707	52,000 to 59,000			0.94	

CHAPTER 4

RIVETED AND WELDED JOINTS

35. Kind of Stress. Riveted joints afford an excellent illustration of tension, compression, and shear, and of the manner of transmission of stress. Figure 57 represents a pair of plates, each of breadth b and thickness t, transmitting a pull P in the direction of their length. The plates are united by means of a pin C, which fits tightly in a hole in the lower plate and passes through a hole in the upper plate. The portion of the upper plate to the left of the pin is in tension. The intensity of this tensile stress is found by dividing the pull P by the area of the *gross*

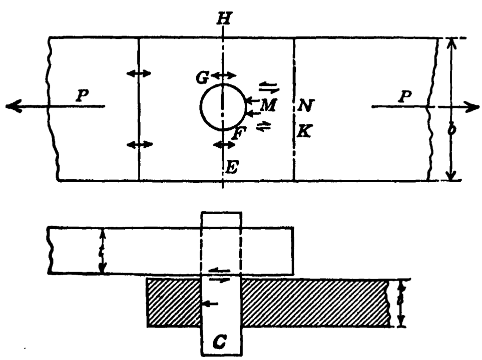


FIG. 57. Stress at a bolted joint.

section bt. At the section EH in the plane of the axis of the hole, the stress is still tension. The area of this *net section* is smaller than that of the gross sections to the left, and the unit stress is greater. If the hole is in the middle of the section and in the line of the pull, half of the total stress is transmitted by the section EF and half by the section GH. The stress which passes EF as tension passes FK as shear. The intensity of this shearing stress in the plate may be calculated by dividing the pull $P/2$ by the section of length FK and the thickness t. At M, the surface of contact of the pin and plate, the stress is compression. The force is transmitted as shearing stress from the portion of the pin in the upper plate to the portion on the lower plate. Finally, the force is transmitted as compression from the lower half of the pin to the portion of the lower plate on the left.

The stress may be regarded as flowing like an electric current, as illustrated in Fig. 58. The circuit is completed through the bodies by which the force is applied.

If the pin in Fig. 57 is slightly smaller than the hole, all the bearing pressure is applied to a narrow strip of the plate at M. The entire shearing stress is then transmitted across two planes which are very close together and which extend from M to the edge of the plate at N. In calculating the shearing stress in the space between a rivet hole and

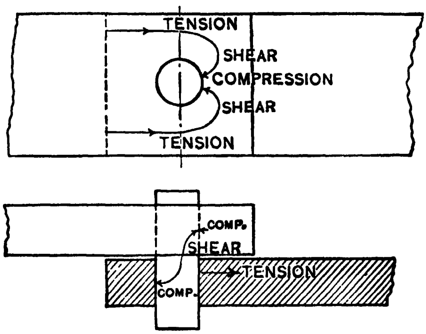


FIG. 58. Flow of stress.

the edge of the plate, it is customary to consider the minimum distance instead of the distance FK.

Problems

35-1. The plates of Fig. 58 are $2\frac{3}{4}$ in. wide and $\frac{3}{8}$ in. thick. The bolt is 1 in. in diameter and exactly fills the hole in each plate. The total pull is 9,240 lb. Find the unit shearing stress in the bolt and the unit tensile stress in the net section and in the gross section of each plate.
Ans. s_s = 11,760 psi; s_t = 8,960 psi in gross section; s_t = 14,080 psi in net section.

35-2. Solve Prob. 35-1 if the plates are $2\frac{1}{2}$ in. wide.

35-3 Solve Prob. 35-1 if the plates are $\frac{5}{16}$ in. thick.

35-4. Solve Prob. 35-1 if the lower plate is 3 in. wide and $\frac{1}{4}$ in. thick.

36. Bearing or Compressive Stress. In calculating the unit bearing or compressive stress at the surface of contact of the pin and plate, it is customary, among engineers, to regard the bearing area as the product of the thickness of the plate multiplied by the diameter of the pin. If d is the diameter of the pin and t is the thickness of the plate, the *bearing area* is td. The bearing area is the projection of that portion of the pin which is inside the plate upon any plane parallel to the axis of the pin.

Figure 59 shows a rectangular bar of thickness d, which is placed across the edge of a plate of thickness t. If the bar crosses the plate at right angles, it is evident that the bearing area is td. If, as in Fig. 60, the bar passes through a hole in the plate, the bearing area is the same; and, if the forces P_1 and P_2 are balanced with respect to the center of the plate, the bearing stress is uniform over the entire area. If the forces are not balanced, the area and the *average* bearing stress remain the same, but the maximum bearing stress is greater. If there is force on the bar on one side of the plate only, the maximum bearing stress is much greater than the average.

Figure 61 shows a round bolt or pin passing through a plate. The actual area of contact is the lower half of the surface of the cylinder of length t and diameter d.

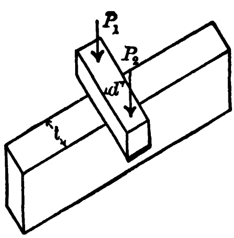


FIG. 59. Bearing.

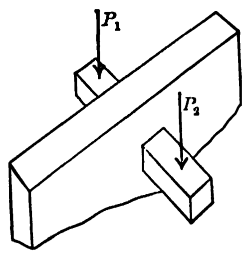


FIG. 60. Bearing.

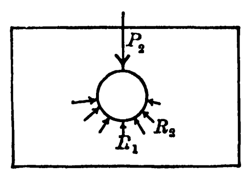


FIG. 61. Bearing.

The reactions R_1, R_2, etc., are not all vertical but are nearly normal to the surface of contact. If, as in the case of liquid pressure, these reactions were exactly normal and of equal intensity, the resultant of their vertical components would be the same as if that unit pressure were exerted on the horizontal projection of this cylindrical surface.

Problems

36-1. What is the compressive stress between the bolt and each plate for Prob. 35-1? *Ans.* $s_c = 24{,}640$ psi.

36-2. Two ½-in. plates, each 3 in. wide, are united by a single 1-in. rivet. The unit shearing stress in the rivet is 10,000 psi. Find the unit bearing stress between the rivet and the plates. Find the unit tensile stress in the net sections and in the gross sections of the plates.
Ans. $s_c = 15{,}708$ psi; $s_t = 7{,}854$ psi in net section.

36-3. Two steel plates, each approximately 4 in. wide and ¾ in. thick, were united by three ⅞-in. rivets in a single row *lengthwise* of the plates (Watertown Arsenal, 1911, p. 129). When tested in tension, the first slipping occurred

at a load of 45,200 lb, and the joint failed by shearing the rivets at 89,200 lb. Find the unit tensile stress in the net section of the plates when slipping occurred and the unit bearing stress between rivets and plates.

Ans. s_t = 19,285 psi at the net section at the rivet farthest from the end of the plate; s_c = 22,960 psi at any rivet.

36-4. In Prob. 36-3, what was the ultimate shearing strength of the rivets? What was the maximum tensile stress at any net section? What was the maximum compressive stress between rivets and plates?

Ans. s_s = 49,448 psi; s_t = 38,062 psi; s_c = 45,300 psi.

36-5. A joint similar to that of Prob. 36-3 had 1⅛-in. rivets. First slipping occurred at 48,900 lb, and the joint failed by fracture of one plate at the outer rivet hole under a load of 128,500 lb. Find the ultimate tensile strength of the material. Find the maximum unit shearing stress and the maximum unit bearing stress.

Ans. Tensile strength = 59,590 psi; s_s = ?; s_c = ?

37. Lap Joint with a Single Row of Rivets. Figure 62 shows a *lap joint* with a single row of rivets. In any riveted joint the distance p from center to center of adjacent rivets in a row is called the *pitch*. In solving problems, it is often convenient to consider a single strip of width equal to the pitch. The problem of a lap joint with a single row of rivets then becomes the same as that of Art. 35. This strip may extend from center to center of adjoining rivets, as is shown between the two lower rivets of Fig. 62. The tension is transmitted by the net section between the rivets, and the shear is equally divided between the upper half of the lower rivet and the lower half of the second rivet. The unit strip may be taken to include one rivet, as is shown at the third rivet from the bottom of Fig. 62. The shear is then carried by the single rivet, while the tension is divided.

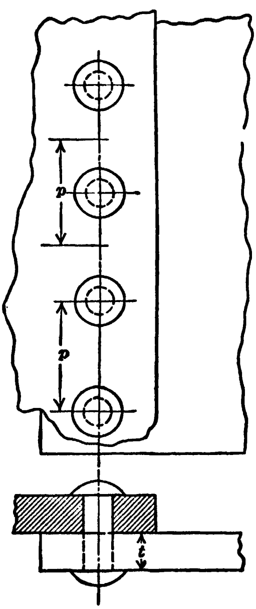


FIG. 62. Single-riveted lap joint.

In the problems, unless otherwise stated, the rivet will be considered as entirely filling the rivet hole. In practice, when rivet holes are punched and not reamed, it is customary to make some allowance for the material near the hole which is weakened by overstrain. This allowance will not be made in the problems which follow. It is assumed that all rivet holes are reamed or drilled, and that the finished rivets fit exactly.

When the width of the plate is given, it is generally best to consider the entire plate as the unit.

Problems

37-1. Two ½-in. plates, each 14 in. wide, are united by five 1-in. rivets in a single row to form a lap joint. The joint transmits a pull of 47,124 lb. Find the unit tensile stress in the gross section and in the net section. Find the unit shearing stress in the rivets and the unit bearing stress between the rivets and the plates.

$$\textit{Ans. } s_t = \frac{47{,}124}{7} = 6{,}732 \text{ psi}$$
$$s_t = \frac{47{,}124}{(14-5)\tfrac{1}{2}} = \frac{94{,}248}{9} = 10{,}472 \text{ psi}$$
$$s_c = \frac{47{,}124}{(5 \times 1 \times \tfrac{1}{2})} = \frac{94{,}248}{5} = 18{,}850 \text{ psi}$$
$$s_s = \frac{47{,}124}{(5 \times 0.7854)} = 12{,}000 \text{ psi}$$

37-2. Solve Prob. 37-1 if the plates are 13 in. wide.

37-3. Two ⅜-in. boiler plates are united by a single row of ⅞-in. rivets to form a lap joint. The pitch is 3⅛ in. The unit stress in the gross section is 6,400 psi. Find the total tension in a unit strip of width equal to the pitch. Find unit tensile stress in net section, unit shearing stress, and unit bearing stress.

$$\textit{Ans. } P = 6{,}400 \times \tfrac{3}{8} \times \tfrac{25}{8} = 7{,}500 \text{ lb}$$
$$s_t = \frac{7{,}500}{(\tfrac{25}{8} - \tfrac{7}{8})\tfrac{3}{8}} = 7{,}500 \times \frac{32}{27} = 8{,}889 \text{ psi}$$
$$s_s = \frac{7{,}500}{0.6013} = 12{,}470 \text{ psi}$$
$$s_c = 7{,}500 \times \frac{64}{21} = \frac{160{,}000}{7} = 22{,}857 \text{ psi}$$

37-4. Solve Prob. 37-3 when the pitch is 3¼ in.

37-5. Solve Prob. 37-3 if the pitch is 3 in. and the unit shearing stress in the rivets is 8,000 psi. *Ans.* s_c = 14,660 psi.

37-6. For a lap joint with a single row of ⅞-in. rivets, what must be the thickness of the plates if the bearing stress is to be twice the shearing stress? *Ans.* t = 0.3436 in.

38. Butt Joint. A butt joint is made when the two principal plates are in the same plane and are united by one or two additional plates which are called *butt straps*. A butt joint with a single butt strap is equivalent to a pair of lap joints placed tandem.

Figure 63 shows a butt joint with double butt straps. Since the rivets are in double shear, the total shear transmitted by each rivet is twice as great as in a lap joint.

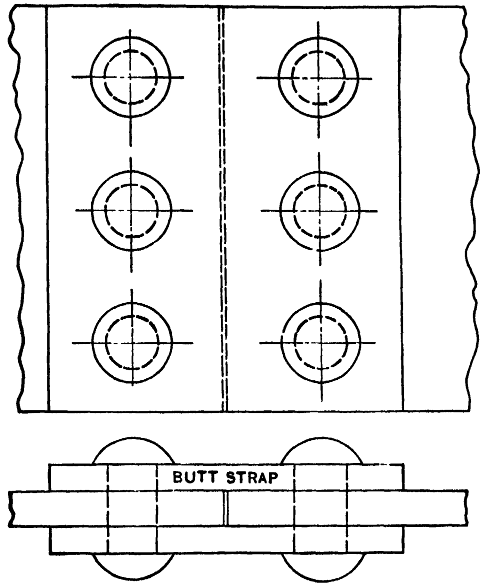


FIG. 63. Single-riveted butt joint.

Example

Two ½-in. plates are united to form a butt joint by two $\frac{5}{16}$-in. butt straps. There is one row of ⅞-in. rivets on each side. If the allowable unit tensile stress in the plate is 10,000 psi and the allowable unit shearing stress in the rivets is 8,000 psi, what should be the pitch?

The area of one rivet is 0.6013 sq in., and each rivet is in double shear. The net cross section which carries the tension equal to the shear in one rivet is $\frac{1}{2}(p - \frac{7}{8})$, so that

$$\frac{1}{2}(p - \tfrac{7}{8})10{,}000 = 2 \times 0.6013 \times 8{,}000$$
$$p - \tfrac{7}{8} \doteq 1.924 \text{ in.}$$
$$p = 2.80 \text{ in.}$$

Problems

38-1. What is the unit bearing stress between rivets and plates, and what is the unit tensile stress in the gross section for the example above?

Ans. $s_c = 6{,}875$ psi; $s_t = ?$

38-2. Two ½-in. plates are united by two ⅜-in. butt straps to form a butt joint with one row of ⅞-in. rivets on each side. The pitch is 2¾ in. Find the unit bearing stress between rivets and plates, the unit tensile stress in the net section, and the unit tensile stress in the gross section when the unit shearing stress in the rivets is 6,000 psi.

Ans. $s_c = 16{,}490$ psi; $s_t = 7{,}696$ psi; $s_t = 5{,}348$ psi.

38-3. Two ⅝-in. plates are united by two ⅜-in. butt straps. There is one row of 1-in. rivets on each side. The pitch is 2¾ in. The unit tensile stress in the gross section is 6,379 psi. Find the unit bearing stress between the rivets

and the plates and between the rivets and the butt straps. Find the unit shearing stress and the unit tensile stress in the net sections.

Ans. s_c = 17,542 psi and 14,618 psi; s_t = 10,025 psi.

Figure 64 is a copy of a photograph of a butt joint tested at Watertown Arsenal. It shows failure by tension across the net section and shear in front of the rivets. It also shows elongation of the rivet holes due to bearing pressure on the plate, combined with shear. The computed tensile stress in the net section of the plate was 67,170 psi.

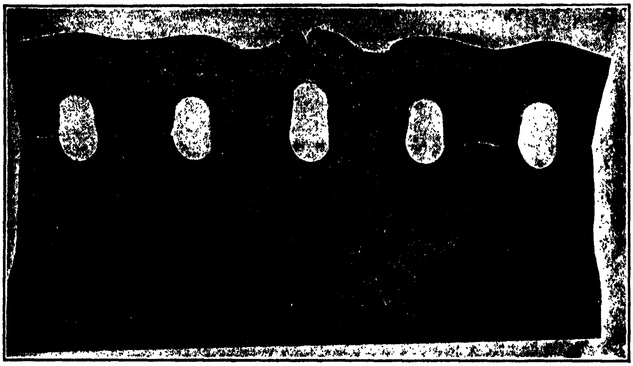


FIG. 64. Failure of riveted plate.

In order to compare the strength of the material in the net section of a riveted joint with the ordinary tension tests, two strips were sheared from each sheet of steel, one lengthwise, the other crosswise of the sheet. These gave as ultimate tensile strengths:

	Psi
Lengthwise	59,180
Crosswise	60,840

The ultimate strength of these test pieces was considerably smaller than the unit stresses in the net sections of the riveted plates which failed in tension. These tension tests show no definite difference between the strength of the strips taken lengthwise of the plate and those taken crosswise of the plate. This is explained by the fact that the plates were rolled in both directions. When rolled metal is worked in one direction only, its tensile strength is greater in that direction.

39. Rivets in More Than One Row. Rivets are frequently arranged in two or more rows. The rivets in the second row may be placed directly behind the rivets in the first row, or they may be arranged zigzag as shown in Fig. 65. Two adjoining rows of zigzag rivets must not be placed too close together or the plate will fail along the diagonal line joining the rivets of the two rows. The Boiler Code of the ASME

gives specifications for the minimum distance between rows of rivets (called *back pitch*).

In computing problems of two or more rows of rivets, it is customary to regard the unit shearing stress the same in all rivets.

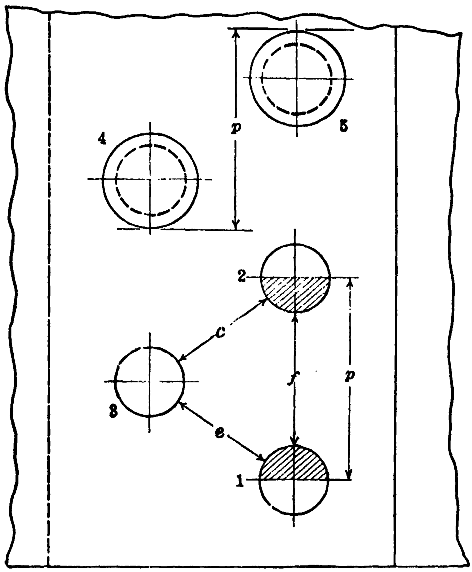


FIG. 65. Double-riveted lap joint.

Where narrow plates are united by several rows of rivets, it is best to take the entire width of the plate as the unit.

Example

Two $\frac{3}{4}$- by 12-in. plates are united to form a lap joint by means of ten 1-in. rivets arranged as shown in Fig. 66. The joint transmits a pull of 60,000 lb. Find

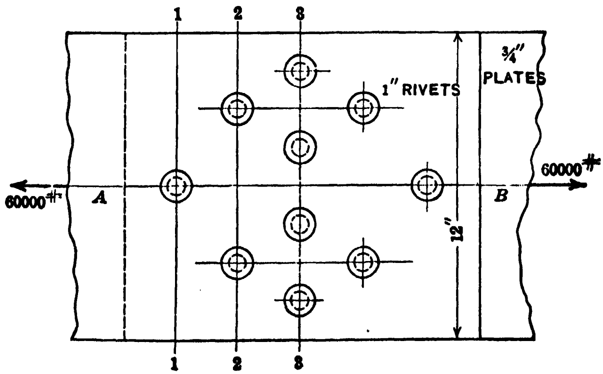


FIG. 66. Multiple-riveted lap joint.

the unit shearing stress in the rivets and the unit tensile stress in plate A at sections 1–1, 2–2, and 3–3.

$$s_s = \frac{60{,}000}{10 \times 0.7854} = 7{,}639 \text{ psi}$$

At section 1–1 the net area is $(12 - 1)\frac{3}{4}$ and

$$s_t = \frac{60{,}000}{11 \times \frac{3}{4}} = 7{,}273 \text{ psi}$$

At section 2–2 the net width is 10 in., but since one-tenth of the total pull has been transmitted by rivet 1 from plate A to plate B, the total tension transmitted through this net section is only 54,000 lb, and

$$s_t = \frac{54{,}000}{10 \times \frac{3}{4}} = 7{,}200 \text{ psi}$$

At section 3–3 the net width is 8 in., but three-tenths of the total pull has been transmitted to plate B through the rivets in sections 1–1 and 2–2 so that the total tension in net section 3–3 is only 42,000 lb.

$$s_t = \frac{42{,}000}{6} = 7{,}000 \text{ psi}$$

Problem

39-1. Solve the foregoing example if the plates are 10 in. wide instead of 12 in.

Psi

Ans. At 1–1, s_t = 8,889
At 2–2, s_t = 9,000
At 3–3, s_t = 9,333

Notice that in Prob. 39-1 the greatest tensile stress is at section 3–3, while in the example the greatest stress is at section 1–1.

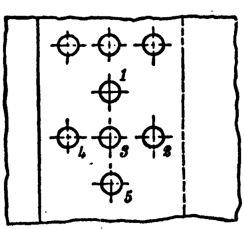


FIG. 67. Rivets in three rows.

In wide plates, such as are used in boilers, it is not convenient to consider the entire width, but it is better to divide the width up into a number of equal units, each of which includes a group of rivets. The width of a *repeating section* is the least common multiple of the several pitches.

Figure 67 shows a lap joint with twice as many rivets in the middle row as in either of the others. The repeating section is taken as equal to the pitch in the outer rows (called the *long pitch*). It includes the two rivets in the middle row and one rivet in each outer row.

Problems

39-2. Two $\frac{3}{8}$-in. plates are united by $\frac{3}{4}$-in. rivets to form a lap joint. The rivets are in three rows as in Fig. 67 with the long pitch 5 in. and the short pitch $2\frac{1}{2}$ in. The unit stress in the rivets is 9,000 psi. Find the unit tensile stress in the gross section of each plate. Find the unit tensile stress in the

net sections at each row of rivets Find the unit compressive stress between rivets and plates.

	Psi
Ans. s_t in gross section	8,483
s_t right upper and left lower	9,979
s_t at middle row	9,088
s_t left upper and right lower	2,495
s_c	14,138

39-3. A butt joint is formed of two ½-in. plates united by two ⅜-in. butt straps. There are two rows of ⅞-in. rivets on each side. The pitch of the inner rows is 3 in. and of the outer rows is 6 in. The unit stress in the gross section of plate is 8,000 psi. Find the unit tensile stress in the net section of the ½-in. plates at each row of rivets. Find the unit stress in the net section at the inner rows of the butt straps. Find the unit shearing stress in the rivets and the unit bearing stress between the rivets and the ½-in. plates.

	Psi
Ans. s_t in ½-in. plates at outer rows	9,366
s_t in ½-in. plates at inner rows	7,529
s_t in butt straps at inner rows	7,529
s_s in all rivets	6,652
s_c between rivets and ½-in. plates	18,286

39-4. In Prob. 39-3, what would be the maximum tensile stress in the butt straps if they were only ¼ in. thick? What should be the minimum thickness of each butt strap in order that the maximum tensile stress at the inner rows should be equal to the maximum tensile stress in the ½-in. plates?

Ans. $s_t = 11{,}294$ psi; $t = 0.301$ in.

39-5. A lap joint is made of two ⅜-in. plates united by five rows of ¾-in. diameter rivets. Referring to Fig. 66 and numbering the rows from left to right, the pitches are 7½, 5, 2½, 5, and 7½ in., respectively. If the bearing stress between rivets and plate is 20,000 psi, find (*a*) the shearing stress in rivets, and (*b*) the tensile stress in each of the five net sections of the top plate.

Ans. (*a*) 12,700 psi; (*b*) 17,800, 16,500, 15,700, 5,880, and 2,220 psi, in the order numbered.

39-6. A lap joint is made of ½-in. plates united by ¾-in. diameter rivets. The rivets are in five rows, numbered as in Fig. 66, the pitches being respectively 6, 4, 3, 4, and 6 in. The unit tensile stress in the gross section is 7,000 psi. Find (*a*) the unit shearing stress in the rivets, (*b*) the unit bearing stress, and (*c*) the unit tensile stress in each of the five net sections of the top plate in the figure.

39-7. A butt joint is made of two ½-in. main plates united by ¾-in. rivets through two 5⁄16-in. butt straps. There are two rows of rivets on each side of the center line The pitch of the outermost rows is 6 in. and of the inner rows 3 in. The tensile stress in the gross section of the main plates is 6,000 psi. Find the unit shearing stress in the rivets, the unit bearing stress on the main plates and on the butt straps, and the unit tensile stress in each net section of the main plates and of the butt straps.

39-8. A butt joint has ½-in. main plates, ⅜-in. butt straps, and ⅞-in. rivets. The rivets are in two rows on each side, the pitch of the outside rows being 7½ in. and of the inner rows 2½ in. The shearing stress in the rivets is

5,000 psi. Find the bearing stress on each plate and the net tensile stress in each row of main and butt plates.

39-9. Figure 68 shows a butt joint with unequal butt straps. The main plates are $\frac{1}{2}$ in. thick, the butt straps $\frac{3}{8}$ in. thick and the rivets $\frac{7}{8}$ in. in diameter.

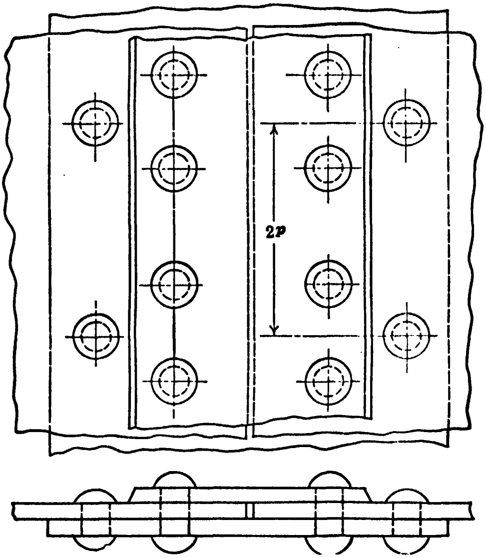


FIG. 68. Double-riveted butt joint.

The pitch is 3 in. for the inner rows and 6 in. for the outer. The stress in the gross section of main plates is 8,000 psi. Find the unit shearing stress in the rivets, the bearing stress on the main plates and on the butt straps, and the tensile stress in the net sections of the main plates and the butt straps.

HINT: The load is distributed so that the shearing stress is constant on all cross sections of the rivets.

		Psi
Ans.	s_t in $\frac{1}{2}$-in. plates at outer rows	9,366
	s_t in $\frac{1}{2}$-in. plates at inner rows	9,035
	s_s in all rivets	7,983
	s_t in lower butt strap at inner rows	9,035
	s_c in $\frac{1}{2}$-in. plates at inner rows	21,940
	s_c in either butt strap	14,630

40. Efficiency of Riveted Joint. The ratio of the strength of a riveted joint to the strength of one of the plates which it unites is called the *efficiency* of the joint. The efficiency may also be defined as the ratio of the smallest allowable load on a repeating section to the allowable load if there were no joint. The calculations may be based upon either the ultimate strengths or the allowable stresses, providing the factors of safety are the same for the different stresses.

To find the efficiency of a riveted joint, the allowable load on the repeating section is calculated in tension at the net sections, in shear of the rivets, in compression between rivets and plates, and in all combinations of these by which failure may occur. The smallest allowable load thus obtained is then divided by the allowable load of the gross section of the unit strip to get the efficiency. When relatively narrow plates are connected by riveting, the strength of the entire plates may be used instead of the repeating section.

The Boiler Code of the ASME gives the following allowable unit stresses for steel:

	Psi
s_t	11,000
s_c	19,000
s_s	8,800

The code further specifies that the strength of a rivet in double shear is twice its strength in single shear.

The AISC specifies the following allowable unit stresses for steel:

	Psi
s_t on net section of plate	20,000
s_s on rivets	15,000
s_c on rivets when in single shear	32,000
s_c on rivets when in double shear	40,000

Example

Two $\frac{3}{8}$- by 12-in. plates are united to form a lap joint by eight $\frac{3}{4}$-in. rivets in two equal rows across the plates. Find the efficiency of the joint, using the entire width of the plates as unit strips, and the allowable stresses of the ASME Boiler Code.

		Lb
Tensile strength of gross section	$12 \times \frac{3}{8} \times 11{,}000 =$	49,500
Tensile strength of net section	$9 \times \frac{3}{8} \times 11{,}000 =$	37,125
Bearing strength of one rivet	$\frac{9}{32} \times 19{,}000 =$	5,340
Shearing strength of one rivet	$0.4418 \times 8{,}800 =$	3,890
Shearing strength of eight rivets		31,120

Since the bearing strength of each rivet is greater than the shearing strength, it is not necessary to calculate the bearing strength of eight rivets. The joint is weakest in shear.

$$\text{Efficiency} = \frac{31{,}120}{49{,}500} = 0.628 = 62.8 \text{ per cent}$$

Problems

(Use ASME Boiler Code in the first four problems.)

40-1. Solve the foregoing example for $\frac{7}{8}$-in. rivets.
Ans. The joint is weakest in tension at net section. Efficiency = 70.8 per cent.

40-2. Two $\frac{11}{32}$-in. plates are united by two rows of $\frac{7}{8}$-in. rivets to form a lap joint. Find the efficiency if the pitch is $3\frac{3}{4}$ in.

		Lb
Strength of gross section.................	=	
Strength of net section...................	=	
Shearing strength of two rivets............	=	
Bearing strength of two rivets............	=	

Ans. Efficiency = 74.6 per cent.

40-3. Solve Prob. 40-2 for a pitch of $3\frac{1}{2}$ in. *Ans.* Efficiency = 75 per cent.

40-4. For what pitch in Prob. 40-2 would the tensile strength of the net section exactly equal the shearing strength?

(*For the next three problems use allowable stresses frequently specified for steel structures:* $s_c = 24{,}000$, $s_t = 18{,}000$, $s_s = 12{,}000$.)

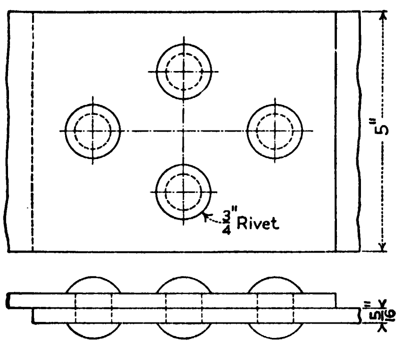


FIG. 69. Triple-riveted lap joint.

40-5. Two $\frac{5}{16}$- by 5-in. plates are united by four $\frac{3}{4}$-in. rivets to form a lap joint. The rivets are arranged in diamond form as shown in Fig. 69. Find the efficiency of the joint.

		Lb
Shear on single rivet..................	$0.4418 \times 12{,}000 =$	5,302
Bearing on single rivet................	$\frac{15}{64} \times 24{,}000 =$	5,625
Shearing strength of four rivets..........................		21,208
Tension net section outer rivets........	$\frac{85}{64} \times 18{,}000 =$	23,906
Tension gross section.................	$\frac{25}{16} \times 18{,}000 =$	28,125
Tension net section at two rivets.......	$\frac{35}{32} \times 18{,}000 =$	19,687
Add shearing strength of outer rivet....................		5,302
Shear outer rivet and tear middle net section..............		24,989

Ans. Efficiency = 75.4 per cent.

40-6. Two $\frac{5}{8}$- by 6-in. plates are united by six 1-in. rivets to form a lap joint as shown in Fig. 70. Find the efficiency.

$$\textit{Ans.}\ \text{Efficiency} = \frac{45{,}000 + 9{,}425}{67{,}500} = 0.806 = 80.6 \text{ per cent.}$$

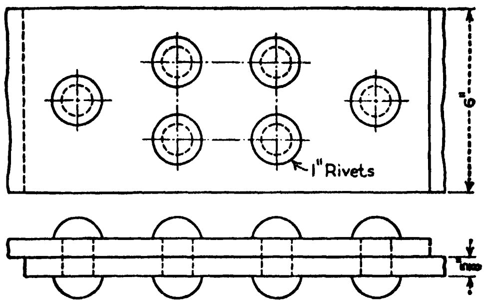


FIG. 70. Quadruple-riveted lap joint.

40-7. Solve Prob. 40-6 for ½-in. plates with all other data unchanged.

	Lb
Allowable load net section outer rivet	45,000
Net section at two rivets plus shear outer rivet	45,425

Ans. Efficiency = 83.3 per cent.

(*Use ASME Boiler Code for the next five problems.*)

40-8. Two ¾-in. plates are united to form a butt joint with two rows of 1-in. rivets on each side. The outer (long) pitch is 6 in. and the inner (short) pitch is 3 in. Find the efficiency.

	Lb	
One rivet, double shear	13,823	
Shear three rivets		41,469
Bearing, rivet on ¾-in. plate	14,250	
Tensile strength of gross section		49,500
Tensile strength net section, outer row		41,250
Net section, inner row	33,000	
One rivet, double shear	13,823	
Total, tension inner row plus shear outer		46,823

Ans. Efficiency = 83.3 per cent.

40-9. Find the minimum thickness of the butt straps of Prob. 40-8.

Ans. $\frac{5 \times \frac{3}{4}}{4} = \frac{15}{16}$ in. total. Use two ½-in. straps.

40-10. Two ¾-in. plates are united to form a butt joint with two rows of 1-in. rivets on each side. The long pitch is 6 in. and the short pitch is 3 in. The butt straps are ½ in. thick. The lower strap (Fig. 68) embraces two rows of rivets on each side of the joint, and the upper narrow strap embraces one row of rivets. Find the strength and efficiency of the joint.

		Lb
Shearing strength all rivets		34,558
Tensile strength net section, outer rows		41,250
Bearing one rivet and ¾-in. plate	14,250	
Bearing two rivets inner row and single shear outer row		35,412
Tension, net section inner row	33,000	

Efficiency = ?

40-11. Two $\frac{3}{4}$-in. plates are united to form a butt joint with three rows of 1-in. rivets on each side. The pitch of the two outer rows is 10 in. and the pitch of the four inner rows is 5 in. The $\frac{1}{2}$-in. butt straps are of equal width. Find the strength and efficiency of the joint.
Ans. Efficiency = 83.8 per cent.

40-12. Solve Prob. 40-11 if the two outer rows on each side have a pitch of $7\frac{1}{2}$ in. and the inner row on each side has a pitch of $3\frac{3}{4}$ in.

40-13. Solve the example (at the beginning of this article) using the allowable stresses of the AISC. *Ans.* Efficiency = 58.9 per cent.

40-14. A butt joint has main plates $\frac{1}{2}$ in. thick and butt straps $\frac{3}{8}$ in. thick. The rivets are $\frac{7}{8}$ in. in diameter and are in two rows on each side. The outer rows have a pitch of 6 in. and the inner rows 3 in. Find the efficiency, using the AISC stresses. *Ans.* 85.4 per cent.

41. Welded Joints. Figure 71 shows two plates connected by a single V arc-welded butt joint, which is reinforced 20 per cent of the plate thickness. A gas-welded joint is similar with a slightly larger angle of the vee and smaller clearance at the bottom. A weld of this kind is generally used in tension. Wide plates welded in this way are cut perpendicular to the weld, milled to a width of 1.5 inches for a length of 9 inches, and loaded to failure on a tension machine as one *qualification test for welders*.

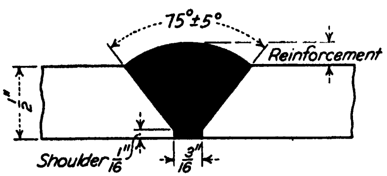


FIG. 71. V-welded butt joint.

Figure 72 shows a fillet weld, which forms a right-angled triangle. The smallest distance from the right angle to the hypotenuse is called the *throat*. When the sides are equal, the throat is the length of either side multiplied by 0.7071.

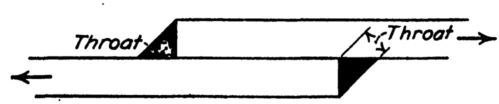


FIG. 72. Fillet welds.

The AISC (1947) specifies the following maximum stresses for welds:

	Psi
Tension and compression on net section through throat of butt welds	20,000
Shear on net section through throat of butt welds	13,000
Shear on net section through throat of fillet welds	13,600

The distinction in shearing stresses for butt and fillet welds is probably an unnecessary refinement to a student whose ambition is to get a thorough grasp of the fundamentals of the subject. Using 13,600

pounds per square inch and considering the throat area, the following allowable values per inch of length are commonly used:

Size of weld, in.	*Lb per inch of length*
1/8	1,200
1/4	2,400
3/8	3,600
1/2	4,800

Figure 73 shows an "end weld." The lower plate exerts shearing force on the lower surface of the fillet. The upper plate exerts tension. The stresses in the weld are combined shear and tension.

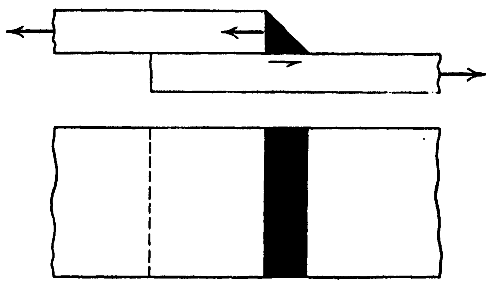


FIG. 73. Fillet end welds.

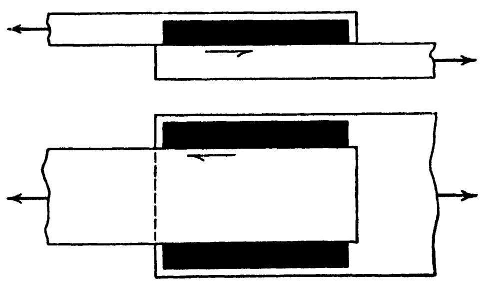


FIG. 74. Fillet side welds.

Figure 74 shows a "side weld." In this figure, the side of the fillet is smaller than the thickness of the plate. Both plates exert shear on the material of a side weld. The allowable shearing stresses for shear at the throat are used for both side welds and end welds.

The joint at the middle in Fig. 75 consisted of two 2- by 3/8-inch plates joined by 3- by 1/4-inch butt straps by means of 1/4-inch welds 2 inches long. Failure occurred in one of the main plates as shown in Fig. 76, under a load of 43,300 pounds, or 57,700 pounds per square inch. Since the welds did not fail, the factor of safety, based on a working stress of 13,600 pounds per square inch, is more than 2.8.

The joint at the right in Fig. 76 consisted of two $2\frac{1}{2}$- by $\frac{3}{8}$-inch plates joined by two $1\frac{1}{2}$- by $\frac{1}{4}$-inch butt straps. Failure occurred at 33,500 pounds by shearing the lower left weld and by shearing the plate

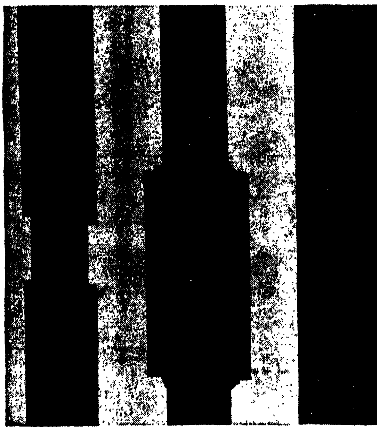


FIG. 75. Welded joints.

FIG. 76. Tested welds.

at the lower right weld. Since the welds were about 1 inch long, the ultimate stress was about 2.46 times the allowable shearing stress for $\frac{1}{4}$-inch fillet welds.

Problems

41-1. Two $\frac{3}{8}$- by 5-in. plates are united by two $\frac{1}{4}$- by 4-in. plates to form a butt joint. The allowable tensile stress in the plates and straps is 20,000 psi. There are 3-in. effective end welds at the ends of each strap. How much side weld is required at each end of each strap?

Ans. 2.41 in. effective at each side.

41-2. Two $\frac{1}{2}$- by 12-in. plates are united to form a butt joint by three 3- by $\frac{3}{8}$-in. butt straps on each side. There are $\frac{3}{8}$-in. fillet welds across the ends of each strap. The allowable tensile stress in the plates and straps is 20,000 psi. How much effective side weld is required at the end of each strap to make the total weld 20 per cent stronger than the plates?

Ans. 1.86 in. of $\frac{3}{8}$-in. fillet on each side.

41-3. Two 5- by $\frac{1}{2}$-in. plates are united to form a lap joint with $4\frac{3}{4}$ in. of effective weld at the ends of the plates. The pull on the plates is 40,000 lb. Are the plates and the welds satisfactory from the standpoint of allowable working stresses?

41-4. Two 6- by $\frac{5}{8}$-in. main plates are united by two 4- by $\frac{3}{8}$-in. butt plates using $\frac{3}{8}$-in. fillet welds. Allowable stress in the plates is 20,000 psi. There is an end weld $3\frac{1}{2}$ in. long on each 4-in. plate. How much side weld is required to develop the strength of the plates?

Ans. 2.42 in. on each side of each plate.

42. Circumferential Stress in Thin Hollow Cylinders. In a thin hollow vessel enclosing a liquid or gas under pressure, the pressure of

the fluid develops stresses in the walls of the vessel. The pressure of a fluid at any point is normal to the surface. The resultant pressure on any portion of a curved surface in any *given direction* is equal to the pressure on a plane surface perpendicular to the given direction and equal in area to the projection of the curved surface upon its plane. Figure 77 represents a portion of the surface of a cylinder of diameter D and of length l perpendicular to the plane of the paper. If P is the

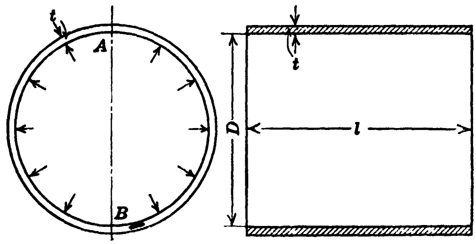


FIG. 77. Hollow cylinder with internal pressure.

pressure on this surface in pounds per square inch, the total pressure on the semicircular surface to the right of the plane AB is $P\pi Dl/2$. The resultant pressure on this surface in the direction normal to AB is PDl, since Dl is the area of the projection of the curved surface upon the vertical plane. There is an equal pressure in the opposite direction upon the curved surface to the left of AB. These two equal and opposite forces are resisted by the circumferential tensile stresses in the sections at A and B. If t is the thickness of the wall of the cylinder, the area in tension is $2tl$, and

$$2tls_t = PDl \tag{42.1}$$

$$s_t = \frac{PD}{2t} \tag{42.2}$$

Note that if the *circumferential* stress is great enough to cause failure, the break will be *longitudinal.*

Problems

42-1. A boiler shell 30 in. in diameter is subjected to a pressure of 300 psi. The allowable tensile stress is 9,000 psi in the gross section. Required: the thickness of the plates. *Ans.* $t = \frac{1}{2}$ in.

42-2. A boiler shell is 5 ft in diameter and $\frac{5}{8}$ in. thick. The efficiency of the longitudinal riveted joint is 80 per cent. If the allowable tensile stress is 11,000 psi, what is the maximum allowable steam pressure? *Ans.* 183 psi.

42-3. A cylindrical tank is 50 in. in diameter and made of $\frac{5}{16}$-in. steel plates. The allowable working stress in the plates is 18,000 psi but there is a longitudinal joint which is 75 per cent efficient. Find the allowable internal pressure.

42-4. A cylinder 50 in. in diameter is made of ½-in. steel plates. The longitudinal joint is a V-welded butt joint as in Fig. 71 for which the allowable tensile stress is 13,000 psi. Find the permissible pressure.

43. Longitudinal Stress in a Thin Hollow Cylinder. The force exerted by the pressure of a liquid or gas in any *given direction* upon a surface is equal to the product of the pressure per unit area multiplied by the projection of the surface upon a plane perpendicular to the given direction. To find the total pressure exerted upon the head of a cylinder in the *direction of the length of the cylinder*, the area of cross section of the cylinder is multiplied by the pressure per unit area. The pressure in the required direction is the same, no matter what may be the form of the cylinder head.

Problems

43-1. A cylinder of 4 ft inside diameter is subjected to an internal pressure of air at 240 psi. What is the total force on the head? *Ans.* 434,290 lb.

43-2. How many ¾-in. bolts with nuts would be required to hold the head of the cylinder of Prob. 43-1. Draw a sketch of the cylinder to show just how the bolts are to be used.

43-3. If the head of the cylinder of Prob. 43-1 is held on by a fillet weld, what size fillet should be used?

If D is the internal diameter of a cylinder, the area of cross section is $A = \pi D^2/4$ and the total pressure longitudinally is $P\pi D^2/4$. The cross section of the cylinder walls which resist longitudinal tension is approximately equal to the inner circumference multiplied by the thickness.

$$s_t \pi D t = \frac{P\pi D^2}{4} \tag{43.1}$$

$$s_t = \frac{PD}{4t} \tag{43.2}$$

Comparison of Eq. (43.2) with Eq. (42.2) shows that longitudinal (or axial) unit stress in a hollow cylinder is one-half as great as the circumferential unit stress.

Equations (43.1) and (43.2) apply also to hollow spheres subjected to internal pressure.

Since the longitudinal tensile stress is only one-half as great as the circumferential unit stress, it is necessary to have the efficiency of the circumferential joints which transmit longitudinal tension only a little greater than one-half the efficiency of the longitudinal joints which resist circumferential stress.

Note that if the longitudinal stress causes a break in the cylinder, the break will be circumferential in direction.

Problems

43-4. A hollow cylinder, 50 in. in diameter, is made of ½-in. plates. The internal pressure is 240 psi. Find the total longitudinal force on one end.

43-5. In Prob. 43-4, what is the approximate area of steel which resists the axial pull? What is the axial unit stress? *Ans.* $A = 78.54$ sq in.; $s_t = ?$

43-6. What is the circumferential unit stress in the cylinder of Prob. 43-5?

43-7. A spherical tank of 6 ft 8 in. inside diameter and 7 ft 0 in. outside diameter is used to transport helium gas at a pressure of 150 atm. What is the approximate average unit stress?

43-8. A cylindrical tank of 6 ft. 8 in. inside diameter and 7 ft. outside diameter is used to transport helium gas. If the ends of this cylinder are hemispheres, what must be its length in order to carry 1.2 times as much gas as the sphere of Prob. 43-7 with the same maximum unit stress?

43-9. A 50-in. diameter cylinder has walls ¼ in. thick. If the internal pressure is 200 psi, find the maximum tensile stress in the walls.

44. Miscellaneous Problems

44-1. Two 3- by 2- by ⅜-in. angles are welded to a plate as shown in Fig. 78. The force P causes a tensile stress of 18,000 psi in the angles. Find the total length of ⅜-in. fillet weld. Find the lengths x and y so that the resultant force P lies at the center of gravity of the angles. *Ans.* $y = 3.0$ in.

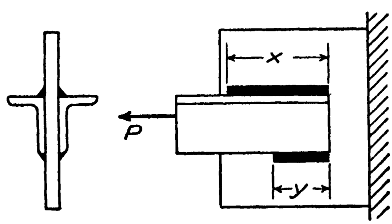


FIG. 78.

44-2. A cylinder is 32 in. in diameter and made of ½-in. plates. The maximum allowable tensile stress is 16,000 psi. Find the maximum allowable pressure.

44-3. A thin cylindrical tank is 5 ft in diameter and has walls ½ in. thick. If the tank resists an internal pressure of 300 psi, find the circumferential and longitudinal stresses. By Mohr's circle, find the maximum shearing stress which occurs on a small element in outside surfaces of the tank. Show your answer on a sketch. *Ans.* 4,500 psi.

44-4. In Prob. 44-3, assume a small element in the cross section of the tank, making one edge of the element coincide with the inside surface of the tank, and find the maximum shearing stress which occurs in this plane. Illustrate your answer with a sketch of the element.

44-5. A 40-in. diameter tube has walls 5/16 in. thick. Sections of the tube are put together with circumferential lap joints consisting of one row of 1-in. rivets spaced on 3-in. centers. Use the ASME Boiler Code allowable stresses and find the permissible internal pressure. *Ans.* 172 psi.

44-6. Solve Prob. 44-5 for ¾-in. rivets. *Ans.* 130 psi.

44-7. A steel pipe has ⅜-in. walls and is 5 ft in diameter. The longitudinal joints are double-riveted butt joints. The rivets are ¾-in. in diameter and are in two rows on each side of the joint. The outer pitch is 5 in. and the inner pitch is 2½ in. The circumferential joints are single-riveted lap joints with the ¾-in. rivets spaced 2½ in. apart. Use the ASME Boiler Code allowable stresses and find the working pressure.

CHAPTER 5

TORSION

45. Torque. A shaft or rod subjected to a pair of equal opposite couples which are in parallel planes at right angles to its length is in *torsion* between these planes. Figure 79 shows a horizontal shaft which is supported by two bearings and carries two pulleys. A rope is wound part of the way around each pulley and fastened to it. Each rope supports a load. The load P on the smaller pulley and part of the reactions of the bearings form a counterclockwise couple. The load W on the larger pulley and a part of the reactions form a clockwise couple. If there is no friction at the bearings, these opposite couples are equal, provided the shaft is stationary or moving in either direction with constant speed. The moment of either couple is the twisting moment or torque in the portion of the shaft between the two pulleys. Torque, which is represented in algebraic equations by T, is expressed in foot-pounds or inch-pounds. In order to distinguish torque and bending moment from work, some writers use *pound-feet* and *pound-inches* for the first two reserve foot-pounds and inch-pounds to mean work or energy. This distinction, however, is not generally made.

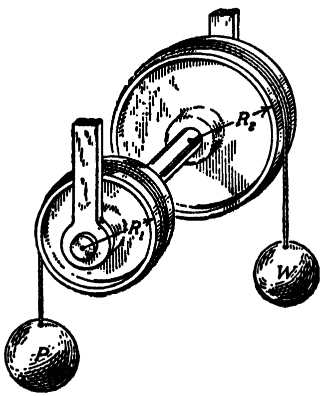


FIG. 79.

Problems

45-1. In Fig. 79 the diameter of the smaller pulley is 40 in. The load P of 600 lb., is hung on a $\frac{3}{4}$-in. rope. Find the torque. *Ans.* $T = 12{,}225$ in.-lb.

45-2. The load W of Fig. 79 is 400 lb and hangs on a 1-in. rope. What is the diameter of the larger pulley if this load balances the load of Prob. 45-1?

45-3. Viewed from either end, what is the direction of the torque in the shaft of Fig. 79 between the pulleys?

45-4. A shaft carries a 5-ft pulley. A belt over this pulley exerts a pull of 1,800 lb on one side and a pull of 400 lb on the other. Find the torque if the thickness of the belt is neglected.

46. Deformation and Stress at Surface of Shaft. Figure 80 represents a shaft fixed at the lower end. DB and EF are lines in its surface

parallel to the axis CO. If the cylindrical surface between the lines DB and EF is developed, it forms the plane rectangle $DBFE$. If a torque is applied to the shaft, twisting it in a counterclockwise direction, the point B is displaced to B' and the point F is displaced to F'. The developed surface $DBFE$ suffers a shearing deformation and becomes the parallelogram $DB'F'E$. Every point on the surface at the upper end is displaced the distance BB'. If a is the radius of the

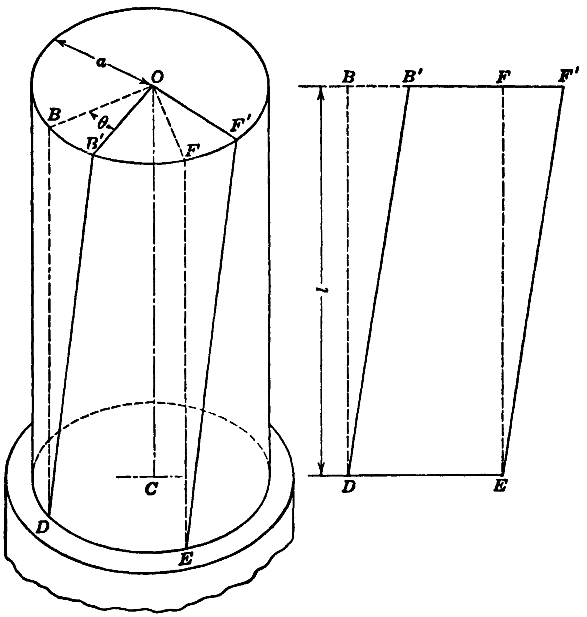


FIG. 80. Portion of shaft in torsion.

cylinder and θ (in radians) is the angle through which the top is turned with reference to the base, the displacement BB' is equal to $a\theta$. If l is the length of the shaft, the unit shearing deformation is given by

$$\gamma = \frac{a\theta}{l} \tag{46.1}$$

and the unit shearing stress in the outer fibers is given by

$$S_s = \frac{Ga\theta}{l} \tag{46.2}$$

Problems

46-1. A 4-in. solid shaft is twisted 3° in a length of 20 ft. What is the unit shearing deformation? *Ans.* $\frac{2}{240} \times \frac{3\pi}{180} = \frac{\pi}{7,200} = 0.000436$

46-2. A $\frac{3}{4}$-in. shaft is twisted 1°43′ in a length of 10 in. Find the unit shearing deformation. (Use tables which give degrees, minutes, and seconds in radians.) *Ans.* 0.001123.

46-3. If the modulus of rigidity in Prob. 46-1 is 11,200,000 psi, find the unit shearing stress at the surface. *Ans.* $S_s = 4{,}883$ psi.

46-4. If the unit stress in Prob. 46-2 is 12,800 psi, what is the modulus of elasticity in shear?

46-5. A shaft 1 in. in diameter is twisted 1°40′ in a length of 10 in. when the shearing stress changed from 3,000 to 20,000 psi. Find G.

46-6. A test piece of low-carbon steel, SAE 1020, was 0.900 in. in diameter. The gage length was 5 in. Some readings were

Torque, in.-lb	*Twist, rad*
0	0
200	0.00138
400	0.00259
600	0.00396
2,000	0.01350
2,200	0.01482
2,400	0.01602
2,600	0.01725

Find the unit deformation at the surface when the torque changes from 200 to 2,200 in.-lb and from 600 to 2,600 in.-lb.

47. Relation of Torque to Angle of Twist. Figure 81 shows the upper end of the shaft of Fig. 80. When the shaft is twisted and the top is turned through an angle of θ radians, any point M, on the elemental ring dA, is moved to M'. Its displacement is $r\theta$ and the unit shearing displacement is given by

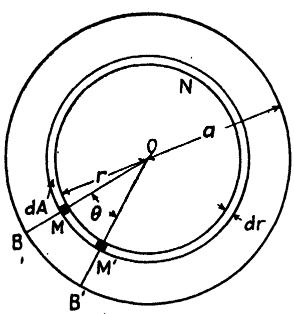


FIG. 81. Shear displacement of torsion.

$$\gamma = \frac{r\theta}{l} \tag{47.1}$$

The unit shearing stress on dA is given by

$$s_s = \frac{Gr\theta}{l} \tag{47.2}$$

It is assumed that any radial line in a circular shaft subjected to torque will remain a straight line. The deformation and hence the stress at all points in the elemental ring will be the same. Deforming this hollow cylinder requires a shearing force equal to the unit stress times the area.

$$\text{Shearing force} = \frac{Gr\theta}{l} \times dA \tag{47.3}$$

The moment of this shearing force with respect to the axis of the cylinder is the product of the force by the distance r.

$$dT = \frac{G\theta}{l} r^2 \, dA \tag{47.4}$$

The entire shaft may be regarded as made up of a series of concentric hollow cylinders, and the total resisting moment, which is equivalent to the external torque, is the integral of Eq. (47.4).

$$T = \frac{G\theta}{l} \int r^2 \, dA \tag{47.5}$$

$$T = \frac{J\theta G}{l} \qquad \text{Formula VIII}$$

where J is the polar moment of inertia of the cylinder and is equal to $\pi a^4/2$.

This theory applies rigidly to circular shafts only. In Fig. 81 the straight line OMB remains straight when the shaft is twisted, provided the sections are circular. When the sections are not circular, a straight line from the center to the surface does not remain straight when torque is applied. The unit stress is not, therefore, necessarily proportional to the distance from the axis, and the equations above are not valid.[1]

Problems

47-1. A 3-in. solid shaft is twisted 2° in a length of 10 ft. Find the torque if $G = 11{,}400{,}000$. *Ans.* $T = 26{,}370$ in.-lb.

47-2. A 1-in. nickel-steel rod is twisted 7° in a 2-ft length by a torque of 500 ft-lb. Find the modulus of rigidity.

47-3. An aluminum bar is 2 in. in diameter and 5 ft long. Find the angle of twist in degrees when a 200 ft-lb torque is applied. *Ans.* 1°19′

47-4. Find the expression for J for a circular shaft with outside radius a and inside radius b (see Fig. 82). *Ans.* $J = \pi(a^4 - b^4)/2$.

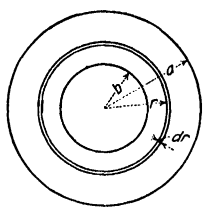


FIG. 82. Circular element of area.

47-5. The rod in Prob. 47-3 is hollow with an outside diameter of 2 in. and an inside diameter of 1 in. Find the angle of twist.

47-6. Find the angle of twist in Prob. 47-3 if the rod is steel.

47-7. A steel shaft for which the modulus of rigidity is 11,200,000 psi is twisted 2°12′ in a length of 10 ft. The shaft is hollow with inside diameter 4 in. and outside diameter 6 in. Find the torque. What would be the torque if the shaft were solid?

[1] St. Venant (1797–1886), mathematician and keen analyst, was the first to solve the torsion problem, but even he could solve very few sections except the circular.

48. Relation of Torque to Shearing Stress. Figure 81 shows that the element dA is displaced a distance $r\theta$ when the shaft is twisted. The displacement is proportional to r. The unit displacement, and, consequently, the unit stress, is proportional to r, which is the distance of the element dA from the axis of the shaft.

If s_1 is the unit shearing stress at unit distance from the axis, the unit shearing stress at a distance r is s_1r. The shearing ~~stress~~ force on an area dA is $s_1r\,dA$ and the resisting moment is $s_1r^2\,dA$. The total moment is given by

$$T = s_1 \int r^2\,dA \qquad (48.1)$$

Since $\int r^2\,dA$ is the polar moment of inertia, which is represented by J,

$$T = s_1J \qquad (48.2)$$

Since s_1 is the unit stress at unit distance from the axis, the unit stress at a distance r from the axis is s_1r, and the unit stress at the surface at a distance a from the axis is given by

$$S_s = s_1a \qquad (48.3)$$

from which

$$s_1 = \frac{S_s}{a} \qquad (48.4)$$

$$S_s = \frac{Ta}{J} \qquad \text{Formula IX}$$

Problems

48-1. A 3-in. solid shaft is twisted by a force of 1,200 lb applied by an arm 4 ft in length, measured from the axis of the shaft. Find the unit shearing stress at the surface. *Ans.* S_s = 10,865 psi.

48-2. A 6-in. solid shaft exerts a torque of 36,000 ft-lb. Find the unit shearing stress at the surface.

48-3. Solve Prob. 48-2 if the shaft is hollow with an inside diameter of 4 in. Find the stress at the inner and outer surface.

Ans. S_s at outer surface = 12,693 psi.
s_s at inner surface = 8,462 psi.

48-4. Solve Prob. 48-2 if the shaft is hollow with an inside diameter of 4 in. and the outside diameter such that the volume is the same as that of a solid 6-in. shaft.

48-5. Find the maximum shearing stress in a 4-in. diameter shaft under a torque of 15,000 ft-lb.

48-6. The shaft in Prob. 48-5 is replaced by a hollow shaft having a 6-in. outside diameter. Find the inside diameter so that the maximum stress is the same as in the original shaft.

49. Relation of Torque to Work. To an arm of length R, measured from the axis of a shaft, a force P is applied which is perpendicular to the plane passing through the axis of the shaft and the point of application of the force. The torque is RP. When the shaft makes one revolution, the point of application of the force moves through a distance $2\pi R$. The work done by the force P is $2\pi RP$. Since PR is the torque,

$$\text{Work} = 2\pi RP = 2\pi T$$

The work per revolution is 2π times the torque. This relation is true whether the torque is due to a single force or to a number of forces.

In problems relating to the work done by a rotating body, solve first for the torque. When this is obtained, it may be used in Formulas VIII and IX.

Problems

49-1. A shaft transmits 600 hp at 240 rpm. Find the work per revolution. Find the torque in foot-pounds.
Ans. 82,500 ft-lb of work; 13,130 ft-lb of torque (or lb-ft).

49-2. What must be the diameter of the shaft of Prob. 49-1 if the allowable unit shearing stress is 6,000 psi? *Ans.* $D = 5\frac{1}{8}$ in.

49-3. The maximum shearing stress in a 4-in. steel shaft is 10,000 psi. If the shaft rotates at 330 rpm, find the horsepower transmitted.

49-4. A shaft transmits 400 hp at 300 rpm under specifications which allow a maximum shearing stress of 9,000 psi. Find the diameter.

49-5. How many horsepower may be transmitted by a hollow shaft which is 4 in. in inside diameter and 10 in. in outside diameter if the allowable shearing stress is 5,000 psi and the speed is 250 rpm? *Ans.* 3,795 hp.

50. Resilience in Torsion. A force P at the end of an arm R twists a shaft of length l through an angle of θ radians. If there is no torque at the beginning, the average force is $P/2$, and the work of twisting is given by

$$\text{Work} = \frac{PR\theta}{2} = \frac{T\theta}{2} = U \tag{50.1}$$

$$U = \frac{T^2 l}{2GJ} \tag{50.2}$$

If the shaft is circular and of radius a,

$$U = \frac{J^2 S_s^2 l}{2a^2 GJ} = \frac{J S_s^2 l}{2a^2 G} \tag{50.3}$$

For a solid circular shaft,

$$U = \frac{\pi a^2 l S_s^2}{4G} = \frac{S_s^2}{4G} \times \text{volume} \tag{50.4}$$

and the energy per unit volume is

$$u = \frac{S_s^2}{4G} \qquad \text{Formula X}$$

Since the modulus of rigidity of metals is much less than the modulus of elasticity in tension or compression, the energy of torsion is relatively large. When Poisson's ratio is $\frac{1}{4}$, G is $2E/5$. When this value of G is substituted in Formula X, the result is

$$u = \frac{5S_s^2}{8E} \tag{50.5}$$

The elastic limit in shear is somewhat smaller than the elastic limit in tension or compression. If the limits were the same, a solid cylinder in torsion would store more energy than a block under direct stress. With direct tension or compression, the force must be very large and the displacement must be very small. With torsion, as in a helical spring, the displacement may be large and the force correspondingly small.

Problems

50-1. An aluminum rod 2 in. in diameter and 5 ft long is twisted by a 200-ft-lb torque. Find the energy stored up per cubic inch and the total resilience.
Ans. $u = 0.146$ in.-lb per cu in.

50-2. A steel rod having a 4-in. diameter is twisted by a 15,000 ft-lb torque. If $G = 12{,}000{,}000$ psi, find the energy per unit volume.

50-3. If the steel shaft in Prob. 49-3 is 20 ft long, find the total energy stored in the shaft while it is transmitting power. Find the angle of twist of the shaft.
Ans. $\theta = 5.73°$.

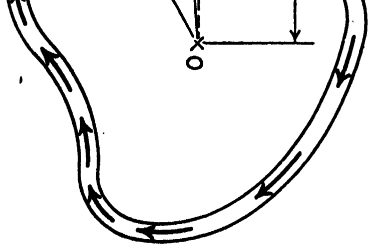


FIG. 83.

51. Thin-walled Closed Sections. When a tubular section is subjected to a torque, the stresses may be computed by a general formula provided the elements of the tube are permitted to warp and are not restrained so that longitudinal stresses are developed. Let Fig. 83 represent the cross section of a hollow closed tube, the small arrows showing the shearing forces which are developed from the torque. The force on the elemental area $t\,ds$ is equal to the unit stress times the area.

$$dF = s_s t\,ds \tag{51.1}$$

The moment of this force may be taken about any point 0 inside the area

$$dT = s_s ty\, ds \tag{51.2}$$

Let $q = s_s t$, where q is called the *shear flow* and is constant for any closed section. This derives its name from the similarity of the flow of stress to the flow of water in pipes. Analogous to the familiar continuity equation,

$$q = s_s t = s_s' t' = s_s'' t'' = \cdots \tag{51.3}$$

since adjacent elements ds which have different thicknesses t must have different stresses in order to satisfy the equations of equilibrium.

For the entire section,

$$T = q \int y\, ds \tag{51.4}$$

The area of the small triangle having a base ds and an altitude y is $\frac{1}{2}y\, ds$. Integrating (51.4),

$$T = 2Aq \qquad \text{Formula XI}$$

where A is the area enclosed within the center line of the tube. After solving for q, the stress at any point may be found from

$$s_s = \frac{q}{t} \tag{51.5}$$

The angle of twist may be found from the energy equations. The energy absorbed in a unit length will be the unit energy from Eq. (24.1) multiplied by the volume (remembering that the length is unity).

$$U = \int \frac{s_s^2}{2G}\, t\, ds \tag{51.6}$$

For one s_s substitute q/t and for the other s_s substitute $T/2At$.

$$U = \int \frac{Tq\, ds}{4GAt} = \frac{T\theta}{2} \tag{51.7}$$

which is obtained by equating the internal energy to the external energy.

$$\theta = \frac{q}{2AG} \int \frac{ds}{t} = \frac{q}{2AG} \sum \frac{p}{t} \tag{51.8}$$

where p is the perimeter, which has a thickness t. If the section has a constant thickness,

$$\theta = \frac{qp}{2AGt} = \frac{s_s p}{2AG} \tag{51.9}$$

Example 1

The thin-walled rectangular section shown in Fig. 84 is 20 in. wide, 10 in. high, and 0.05 in. thick. Find the stresses in the walls when the applied torque is 200 ft-lb. Find the angle of twist for aluminum.

$$A = 200 \text{ sq in.} \qquad p = 60 \text{ in.} \qquad T = 2{,}400 \text{ in.-lb}$$

$$q = \frac{T}{2A} = \frac{2{,}400}{400} = 6 \text{ lb per in.} \qquad s_s = \frac{q}{t} = \frac{6}{0.05} = 120 \text{ psi}$$

$$\theta = \frac{qp}{2AGt} = \frac{6 \times 60}{2 \times 200 \times 4{,}000{,}000 \times 0.05} = 0.000{,}0045 \text{ rad per inch of length}$$

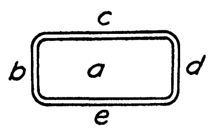


FIG. 84.

Example 2

In Example 1, let the horizontal members be 0.04 in. thick and the vertical members 0.05 in. Solve.

Letter by Bow's notation.

$$A = 200 \text{ sq in.} \qquad T = 2{,}400 \text{ in.-lb}$$

$$\left(\frac{p}{t}\right)_{ab} = \left(\frac{p}{t}\right)_{ad} = \frac{20}{0.04} = 500 \qquad \left(\frac{p}{t}\right)_{ae} = \left(\frac{p}{t}\right)_{ac} = \frac{10}{0.05} = 200$$

$$q = \frac{2{,}400}{400} = 6 \text{ lb per in.} \qquad s_s = \frac{6}{0.05} = 120 \text{ psi for } ab \text{ and } ad$$

$$s_s = \frac{6}{0.04} = 150 \text{ psi for } ac \text{ and } ae$$

$$\theta = \frac{q}{2AG}\sum\frac{p}{t} = \frac{6(400 + 1{,}000)}{2 \times 200 \times 4{,}000{,}000} = 0.000{,}00525 \text{ rad per inch of length}$$

Problems

51-1. A thin cylinder is 20 in. in outside diameter and has walls 0.2 in. thick. Find the shearing stress in the walls when a 500 ft-lb torque is applied. Check your answer by Formula IX. *Ans.* 48.7 psi.

51-2. An airplane section is nearly elliptical in shape. The walls are 0.04 in. thick on the top half and 0.05 in. thick on the bottom half. The total perimeter is 50 in. and the area 150 sq in. Find the shearing stresses in the walls when the torque is 3,000 in.-lb.

52. Helical Springs. An interesting illustration of torsion is the elongation or compression of a helical spring, such as is shown in Fig. 85. A helical spring is made by winding a wire or rod on a cylinder (in a single layer, usually). The radius of the coil of the spring is the sum of the radii of the wire and the cylinder about which it is wound. When the spring is to be used in tension, the ends of the wire are turned

in to the center, in order that the force may be applied in the line of the axis. Figure 85,II is a plan of the lower turn. The force P is normal to the plane of the paper. Any portion of the spring CBO may be considered as a free body. The section at O is perpendicular to the wire. The plane of this section passes through the center C. The force P at C has no bending moment with respect to the section at O. The effect of the force P acting on the arm CBO is independent of the form of the arm. As far as the stresses at O are concerned, CBO might be a straight rod from C to O. The effect of the force P on the section at O is a torque $P \times R$. Since O is any point on the helix, the entire wire, except the portion CB and a similar portion at the top, is subjected to the torque $P \times R$. In addition to this torsion, there is a constant total shear P. Since the coils are not exactly horizontal, there is another slight correction. Both of these, however, are neglected in ordinary calculations.

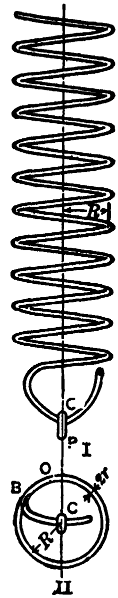


FIG. 85. Helical spring.

The total elongation of a helical spring is calculated by multiplying the angle of twist in the entire length of the wire by the radius of the coil.

Problems

52-1. A rod 0.2 in. in diameter is used to make a helical spring of 20 turns. The radius of the coil from the axis to the center of all sections is 1 in. What is the elongation, due to a load of 3 lb, if the modulus of rigidity is 12,000,000 psi?

$$T = 3 \text{ in.-lb} \qquad J = \frac{0.0001\pi}{2} \qquad \text{length of rod} = 40\pi$$

$$\theta = \frac{3 \times 40\pi \times 2}{12{,}000{,}000 \times 0.0001\pi} = 0.2 \text{ rad}$$

Ans. Elongation $= 0.2 \times 1 = 0.2$ in.

52-2. What is the unit shearing stress in Prob. 52-1? *Ans.* $S_s = 6{,}000/\pi = 1{,}910$ psi.

52-3. If the same rod were used to make a spring of 10 turns, each of 2-in. radius, what would be the elongation due to a load of 3 lb, and what would be the unit shearing stress? *Ans.* 0.8 in.; 3,819 psi.

52-4. At Watertown Arsenal, a steel rod 1.24 in. in diameter and about 241 in. long was formed into a helical spring of 7.64 in. *outside* diameter. A load of 5,000 lb shortened this spring 4.64 in. Find the modulus of shearing elasticity. *Ans.* 11,460,000 psi.

52-5. In Prob. 52-4, find the unit shearing stress under the load of 5,000 lb. *Ans.* 42,740 psi.

52-6. If R is the radius of the helix, r the radius of the rod, P the load, G the modulus of elasticity in shear, and n the number of turns, prove that

$$\text{Elongation} = \frac{4PR^3n}{Gr^4}$$

52-7. If S_s is the allowable unit shearing stress, find the elongation of a spring in terms of S_s, G, R, r, and n. *Ans.* Elongation $= 2\pi S_s R^2 n/Gr$.

52-8. Find the expression for the elongation of a helical spring in terms of S_s, G, R, r, and l, in which l is the length of the rod. *Ans.* Elongation $= S_s Rl/Gr$.

53. Torsional Stresses beyond the Elastic Limit. When a rod is twisted, the stress-strain diagram may be plotted as in a tension test.

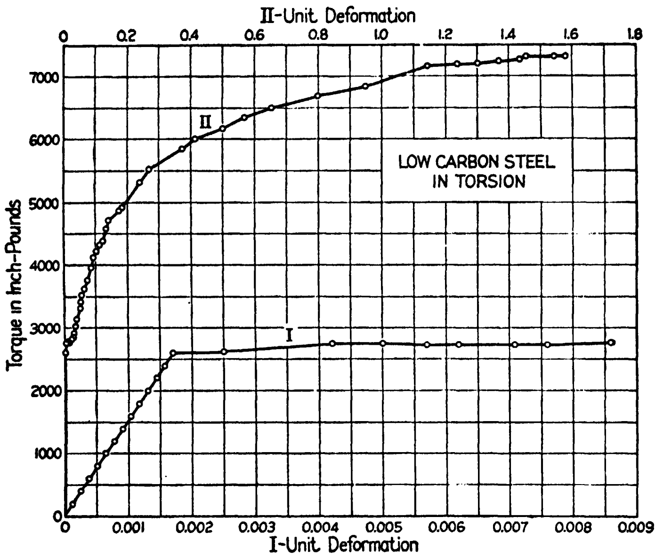


FIG. 86. Torque-strain diagram.

It is more common to plot the torque–angle-of-twist or the torque-strain diagram. Figure 86 shows the surface deformation plotted against the torque for an SAE 1020 hollow steel rod which had an outside diameter of 0.900 inch and an inside diameter of 0.500 inch. The diagram shows an elastic limit at 2,600 inch-pounds torque (corresponding to an outer fiber stress of 20,000 pounds per square inch) and a yield point at approximately 2,700 inch-pounds (corresponding to a stress of 20,800 pounds per square inch). Beyond the elastic limit, Formulas VIII, IX, and X do not hold. It is customary, however, to use Formula IX when comparing similar tests of the same material in order to get some kind of a comparison. For brittle mate-

rials which have no yield point, the formula gives good comparative values.

If the ultimate torque is substituted for T in Formula IX and the equation solved for S_s, the result has frequently been called the *modulus of rupture.*

$$S_{sr} = \frac{T_m a}{J} \qquad (53.1)$$

Note that the modulus of rupture does not give the true stress, and that the formula can be used only for round rods to get a comparison of a fictitious stress at failure of (usually) brittle materials.

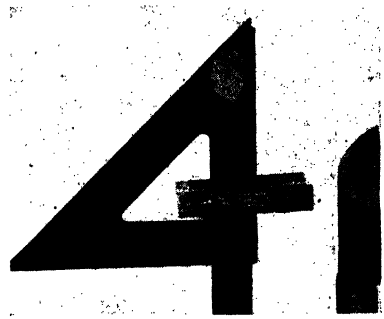


FIG. 87. Torsion failure of cast iron.

Figure 87 shows a cast-iron bar which was tested in torsion. Failure took place at right angles with the direction of the resultant tensile stress which is caused by shearing stress. The 45° triangle shows how accurately the experiment agrees with the theory.

After a hollow rod of mild steel in torsion has passed the yield point practically all fibers have nearly the same stress.

If $\bar{s}_s$ is a shearing stress which is constant throughout the section, the total stress on a circular element of radius r and thickness dr (Fig. 82) is $\bar{s}_s 2\pi r\, dr$. The resisting torque of the stress on this element is the total force multiplied by r.

$$T = 2\pi \bar{s}_s \int_b^a r^2\, dr = \frac{2\pi \bar{s}_s}{3}(a^3 - b^3) \qquad (53.2)$$

$$\bar{s}_s = \frac{3T}{2\pi(a^3 - b^3)} \qquad (53.3)$$

For a solid cylinder,

$$\bar{s}_s = \frac{3T}{2\pi a^3} \qquad (53.4)$$

A solid cylinder does not differ so greatly from a hollow cylinder. Since the torque varies as the fourth power of the diameter when the force varies as the distance from the axis, a hollow cylinder with inside radius one-half the outside radius has fifteen-sixteenths as great torque as a solid cylinder. If a small portion of a solid cylinder near the axis has not been deformed to the yield point, the error is not great if it is assumed that the stress is constant throughout. Equation (53.3) may be used with confidence in the determination of the ultimate shearing strength of soft steel in torsion if the gage length has an inside diameter which is three-fourths as great as the outside diameter. If, however, the material has been strained nearly to failure and then permitted to stand for some time, the stress on reloading may be proportional to the distance from the axis.

Problems

53-1. The hollow cylinder of Fig. 86 had an outside diameter of 0.900 in. and an inside diameter of 0.500 in. Find the shearing stress by Eq. (53.3) when torque was 2,750 in.-lb. Check by Formula IX. *Ans.* s_s = 17,390 psi.

53-2. Reading the maximum torque from Fig. 86, compute the modulus of rupture for the hollow rod of Prob. 53-1. Find also the stress from Eq. (53.4). *Ans.* At T = 7,300 in. lb, s_s = 46,200 psi.

53-3. A piece of the rod from the sample used in Fig. 86 was turned to a diameter of ¾ in. and tested in double shear with the apparatus of Fig. 56. The maximum load was 38,800 lb. Find the ultimate shearing stress and compare with Prob. 53-2.

53-4. Referring to the cast-iron rod in Fig. 87, sketch an element in the *front surface* of a vertical shaft, with four arrows representing the shear forces acting on this element from the adjacent material when the applied torque is counterclockwise, viewed from either end. Draw lines on this element to show the direction of the maximum tension and the direction of the line of rupture which would result from this tension. Compare with Fig. 87.

53-5. The rod of Fig. 87 was 1.24 in. in diameter and failed under a torque of 9,010 in.-lb. Find the modulus of rupture. Find the ultimate tensile strength. *Ans.* $s_t = s_s$ = 24,070 psi.

54. Miscellaneous Problems

54-1. A steel rod 1 in. in diameter twisted 1°30′ in a 10-in. gage length when a 3,000 in.-lb torque was applied. Find the modulus of rigidity.

54-2. Find the total energy stored up in the shaft in Prob. 54-1 when the 3,000 in.-lb torque was applied.

54-3. Two sections of a shaft transmitting power are joined by a flange coupling. Six 1-in. diameter bolts are used to connect the flanges. The bolts are spaced around the flanges on an 8-in. diameter circle. If the maximum shearing stress of the shaft is the same as the bolts, find the minimum size of shaft so that the bolts will reach the allowable stress first.

54-4. A steel shaft 3 in. in diameter is welded to a hub by means of a circumferential ⅜-in. fillet weld where the shaft joins the hub. If the weld is stressed to the allowable working stress, find the maximum stress in the shaft.

54-5. A 2-in. diameter steel shaft is welded to a hollow steel shaft which has an outside diameter of 4 in. and an inside diameter of 2 in. The longitudinal axes of the shafts are collinear so that the solid 2-in. shaft can transmit power to the hollow shaft. The shafts are connected by running a ⅜-in. fillet weld around the solid shaft where it butts against the hollow shaft. If the maximum shearing stress in the solid shaft is 10,000 psi, find (*a*) the horsepower transmitted; (*b*) the stress in the throat of the weld; and (*c*) the shearing stress in the outer fibers of the hollow shaft.

54-6. A hollow shaft of 4 in. outside diameter and 2 in. inside diameter is 10 ft long and gripped at both ends. A 9,000 ft-lb torque is applied at the middle of the shaft. Find the maximum shearing stress in the shaft.

54-7. If S_s is the allowable unit shearing stress, N is the number of revolutions per minute, hp is the horsepower, and a is the radius of a solid shaft, show that

$$a^3 = \frac{33{,}000 \times 12 \times \text{hp}}{\pi^2 \times N \times S_s} = \frac{40{,}123 \text{ hp}}{N \times S_s}$$

54-8. If the allowable unit shearing stress is 5,000 psi, show that the diameter of a solid shaft should be approximately $d = 4\sqrt[3]{\text{hp}/N}$.

54-9. In Prob. 52-4, find the total external work done on the spring using the load and shortening. Divide by the volume to get the average work per cubic inch. *Ans.* 11,600 in.-lb; 39.8 in.-lb per cu in.

54-10. Check Prob. 54-9, using internal energy (Formula X) and the answers to Probs. 52-4 and 52-5.

54-11. A spring at the Watertown Arsenal was made of 36 lb of steel rod 1.02 in. in diameter. The outside diameter of the coil was 4.30 in. A load of 11,000 lb changed the length of this spring from 20.63 to 16.67 in. After the load was removed, the spring returned to its original length to within 0.02 in. Find the energy per cubic inch and the energy per pound. *Ans.* 50.4 ft-lb per lb.

54-12. In Prob. 54-11, what was the maximum shearing stress due to torsion? *Ans.* 86,580 psi.

CHAPTER 6

BEAMS

55. Definition. A beam is a rigid body subjected to parallel transverse forces. Figure 88 is the front view of a horizontal beam, which is firmly held at the right end and carries a load P at the left end. The wall in which the beam is fixed exerts an upward pressure R_1 and a downward pressure R_2. These two *reactions* and the load P constitute the set of parallel transverse forces. Figure 89 shows a second beam,

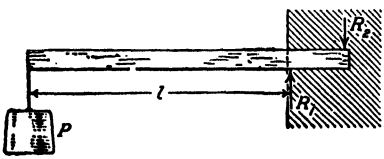


FIG. 88.

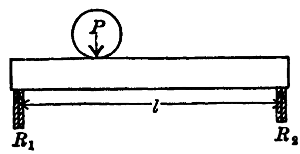


FIG. 89. Beam supported at ends.

which is *simply supported* near the ends and carries a concentrated load P between the supports. In addition to the concentrated load P and the reactions R_1 and R_2 in Figs. 88 and 89, the weight of the beam itself furnishes another parallel transverse force. If the material of the beam is of uniform density and all cross sections are alike, the weight is *uniformly distributed.*

Figures 88 and 90 represent beams which are fixed at one end and free at the other. A beam supported in this way is called a *cantilever.*

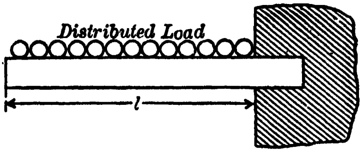


FIG. 90. Cantilever.

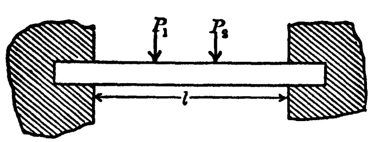


FIG. 91. Beam fixed at both ends.

Figure 91 shows a beam *fixed* at both ends. The beam of Fig. 92 is *fixed* at the right end and *supported* at the left end. Figure 93 shows a *simply supported* beam which *overhangs* its supports. A beam with three or more supports, as in Fig. 94, is a *continuous beam.*

Beams may be classified also in accordance with the type of loading.

Figures 88, 89, and 92 show a single *concentrated* load on each beam. Figure 90 shows a load which is *uniformly distributed* over the entire length of the beam. The beam of Fig. 94 carries a uniformly distributed load on the left overhang and on one-half of the left span and carries another distributed load of greater weight per unit length over the right span. The beam of Fig. 93 is loaded uniformly for the left half of its length and has a concentrated load on the right end. Figure 91 shows two concentrated loads, symmetrically placed.

A beam is not necessarily horizontal. A vertical fence post subjected to the horizontal force of the wind or the horizontal reactions of

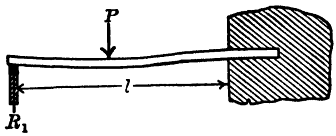


FIG. 92. Beam fixed at one end and supported at the other.

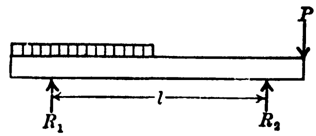


FIG. 93. Beam overhanging its supports.

the hinges of a gate is a cantilever beam. A post at the end of a line of wire fence is a vertical beam which is supported horizontally at the top and partially fixed at the bottom and carries a horizontal load at each wire.

56. Reactions at Supports. The calculation of the reactions at the supports of a beam is a problem of the equilibrium of nonconcurrent, coplanar forces. The general problem of nonconcurrent, coplanar forces has three unknown quantities and requires three independent equations. When all the forces are parallel, as in most cases of beams, there are only two unknowns, and only two independent equations are required. One of these equations *must* be a *moment* equation; the other *may* be either a *moment* equation or a *resolution* equation. It is best to solve by two moment equations and then check by a resolution equation. In order to eliminate one unknown, the origin of moments for the first equation, at least, should lie on one of the unknown reactions.

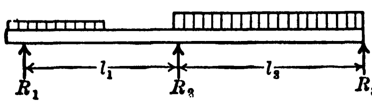


FIG. 94. Continuous beam.

Example

A beam 20 ft long weighing 30 lb per ft is supported at the right end and 4 ft from the left end and carries a load of 80 lb at the left end (Fig. 95). Find the reactions and check.

The total weight of the beam is $30 \times 20 = 600$ lb. Taking moments about an axis through the right support:

$$\begin{aligned} 600 \times 10 &= 6{,}000 \\ 80 \times 20 &= 1{,}600 \\ \hline R_1 \times 16 &= 7{,}600 \\ R_1 &= 475 \text{ lb} \end{aligned}$$

Taking moments about an axis through the left support:

$$\begin{aligned} 600 \times 6 &= 3{,}600 \\ 80 \times -4 &= -320 \\ \hline R_2 \times 16 &= 3{,}280 \\ R_2 &= 205 \text{ lb} \end{aligned}$$

It will be noticed that in the first part of the example counterclockwise moment is written positively and in the second part clockwise moment is written positively. This is done for convenience to avoid negative signs as much as possible. It makes no difference which sign

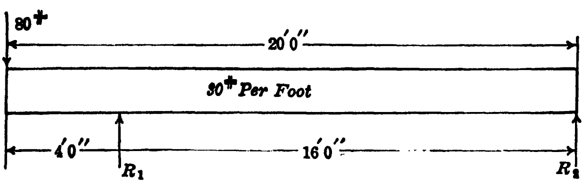


FIG. 95. Beam overhanging left support.

is given to a moment expression, provided the same convention is retained throughout one equation. When the moments are not all of the same sign, it is convenient to take as positive the rotation which has the greatest number of terms. The direction of a moment should always be determined by noting which way it tends to rotate about the axis of moments rather than by observing the mathematical sign of the forces and the arms. Again, some would write these moments in the form of an equation, the first part of the second example being

$$600 \times 10 + 80 \times 20 = 16R_1$$

This is sometimes convenient when there are factors which can be canceled, but generally it is better to arrange the work as shown in the example. Where there are a large number of terms, several of which are negative, it is advisable to put the positive moments in one column and the negative moments in another.

Problems

(Always make a sketch of the beam showing all dimensions and loads in the solution of problems and examples. Even if there is a sketch in the book, make your own sketch. The checker should always be able to follow the calculations and sketches of the designer.)

56-1. A horizontal beam is 18 ft long and is supported at the right end and at 3 ft from the left end. It carries a uniformly distributed load of 250 lb per ft including its own weight and a 900-lb concentrated load at 5 ft from the right end. Find the reactions. *Ans.* $R_L = 3{,}000$ lb.

56-2. A horizontal beam 20 ft long is supported at the right end and at 4 ft from the left end. It carries a uniformly distributed load (which includes the beam's own weight) of 320 lb per ft and two concentrated loads: 800 lb on the left end, and 4,800 lb at 4 ft from the right end. Find the reactions. *Ans.* $R_R = 5{,}800$ lb.

56-3. A horizontal beam 24 ft long is supported 4 ft from the left end and 2 ft from the right end. It carries a uniformly distributed load, including its own weight, of 360 lb per ft. A concentrated load of 1,200 lb is 1 ft from the left end, and a load of 2,160 lb is 7 ft from the right end. Find the reactions and check. *Ans.* $R_L = 6{,}800$; $R_R = 5{,}200$.

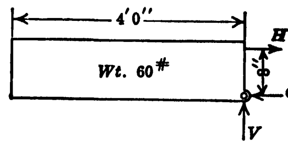


FIG. 96. Beam supported by horizontal couple.

56-4. A beam 4 ft long weighing 60 lb, with its center of gravity at the middle, is hinged at the lower right corner (Fig. 96) and held horizontal by a horizontal pull 8 in. above the hinge. Find this horizontal pull (H), the horizontal component of the hinge reaction (C), and the vertical component of the hinge reaction (V). *Ans.* H, 180 lb; C, 180 lb; V, 60 lb.

57. Shear in Beams. Figure 97 represents a cantilever which is fixed at the right end and carries a concentrated load P near the left

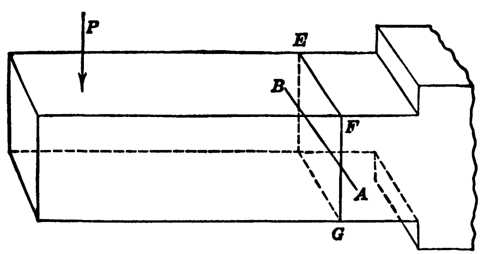


FIG. 97. Section of cantilever.

end. A section EFG across this beam separates the left portion as a free body. (Figure 99 shows the front elevation of this cantilever.) The load P is at a distance a from the left end, and the section EFG is at a distance x from the left end. The weight per unit length is w. When the portion of the beam to the left of the section EFG is considered as the free body in equilibrium, the *external* forces are the load P and the weight of the portion, which is wx. This portion of the beam is kept in equilibrium by the *internal* forces which the portion on the right of the section EFG exerts across the section.

Figure 98 shows the beam actually cut in two at the section FG. A

cylinder, with its axis horizontal, perpendicular to the length of the beam, separates the two portions near the bottom, and a short horizontal chain connects them near the top. A vertical chain is attached to the right end of the left portion.

The tension in each chain and the compression in the cylinder are calculated as a problem of the equilibrium of nonconcurrent, coplanar forces. By a vertical resolution, the tension in the vertical chain is shown to be equal to the sum of the two vertical loads. The horizontal resolution shows that the pull H of the horizontal chain is equal to the horizontal push C of the cylinder.

If the section in Fig. 99 is considered to be glued, then all the glue will be in shear, the upper half will also be in tension, and the lower half will be in compression.

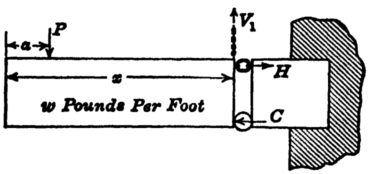


FIG. 98. Cantilever shear and tension.

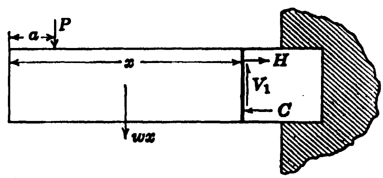


FIG. 99. Resisting shear and moment at glued section.

The vertical shear, V_1 of Fig. 99, is called the *resisting shear.* The resultant of all the forces parallel to the section which act on the portion of the beam on either side of the section is called the *external shear.* In a horizontal beam the external shear (for a section at right angles to the beam) is vertical and is called the *total vertical shear.* In formulas total vertical shear is represented by V. The resisting shear on one side of any section is equal and opposite to the external shear acting on the portion of the beam on the other side of the section. The external shear on the portion of the beam to the left of the section is $P + wx$ acting downward and is equal to the resisting shear with which the portion to the right of the section acts on the glue. Since the entire beam is in equilibrium under the action of the external forces, the external shear on the portion to the right of the section must be equal and opposite to the shear on the left portion. In like manner, the portion to the left of the section exerts a shear equal and opposite to V_1 upon the portion to the right.

The *magnitude* of the vertical shear may be determined from the vertical resolution of all the external forces which act on either the left or the right portion of the beam. The *sign* of the shear is regarded as positive when the resultant of all the vertical forces which act on

the portion to the left of the section is upward. In Figs. 97 to 99, inclusive, the forces P and wx are downward. The vertical shear, therefore, is negative. In Fig. 99, the single-barbed arrow to the right of the section is upward. This arrow represents the *resisting shear* across the section FG with which the portion to the right of the section opposes the external shear on the portion which is taken as the free body.

Example

A uniform horizontal beam 10 ft long, weighing 12 lb per ft, is supported at the ends and carries a load of 30 lb 3 ft from the left end (Fig. 100). Find the total vertical shear at a section 2 ft from the left end and at a section 4 ft from the left end.

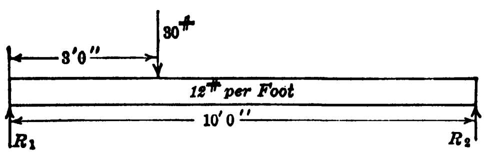


FIG. 100. Beam supported at ends.

The reactions are 81 lb at the left support and 69 lb at the right support. With the left portion used as the free body in equilibrium, at 2 ft from the left end,

$$V_2 = 81 - 2 \times 12 = 57 \text{ lb}$$

At 4 ft from the left end,

$$V_4 = 81 - 4 \times 12 - 30 = 3 \text{ lb}$$

Just before the load of 30 lb is reached, the shear is

$$V_{3-} = V_2 - 12 = 57 - 12 = 45 \text{ lb}$$

Just after the load is passed, the shear is

$$V_{3+} = 45 - 30 = 15 \text{ lb}$$

Problems

57-1. Check the example by using the right portion of the beam as the free body. The numbers should be the same as in the example with the signs opposite.
57-2. In Prob. 56-2, find the shear at 2, 4, 6, and 10 ft from the left end. Check.
57-3. In Prob. 56-3, find the shear at 6, 12, and 18 ft from the left end.

It is often convenient to represent the total shear at all sections of a beam by means of a diagram. Figure 101 is the *shear diagram* for a uniform horizontal beam which is 12 feet long, weighs 20 pounds per foot, and is supported at the ends. Each end reaction is 120 pounds. At any section of the beam, the shear is the algebraic sum of the left reaction upward and the weight of the portion of the beam between

the left end and the section acting downward. Infinitely near the left support, the weight of the portion of beam to the left is negligible. The shear, therefore, is the left reaction of 120 pounds. Infinitely near the right support, the shear is the reaction of 120 pounds minus the weight of practically all the beam, which is 240 pounds. The shear is, therefore, minus 120 pounds. At 1 foot from the left end, the shear is

$$V_1 = 120 - 20 = 100 \text{ lb}$$

The equation for the shear is

$$V_0^{12} = 120 - 20x \tag{57.1}$$

The limits on V indicate that the equation is valid for values of x between 0 and 12 feet. Since this is an equation of a straight line, it is necessary to know only the end points of the shear ordinates.

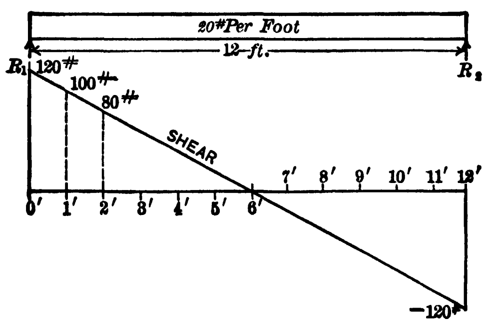


FIG. 101. Shear diagram for distributed load.

Figure 102 is the shear diagram for a beam which is 10 feet long, weighs 60 pounds per foot, is simply supported at the ends, and carries a load of 200 pounds 3 feet from the left end. By moments about the right support, the left reaction R_1 is found to be 440 pounds. By moments about the left support, the right reaction is found to be 360 pounds. The sum of these reactions is 800 pounds, which checks the total load.

The shear is 440 pounds infinitely near the left support. It drops 180 pounds in the first 3 feet and is 260 pounds infinitely close to the left of the load of 200 pounds. Under this load, the shear diagram drops vertically 200 pounds. The shear infinitely close to the right of the 200-pound load is 60 pounds. Beyond the concentrated load, the shear drops at the rate of 60 pounds per foot for the remaining 7 feet. It is minus 360 pounds infinitely close to the left side of the right support. The right reaction of 360 pounds raises the diagram to the

initial line. The diagram crosses the initial line, or zero ordinate, 1 foot to the right of the concentrated load, which is 4 feet from the left support.

The shear diagram of Fig. 102 is a vertical straight line at each support and at the concentrated load, and the discussion refers to points infinitely near the supports or the load. This method of treatment assumes that the loads and reactions act on mathematical lines. In reality, the *surface* of contact is a band of some width extending across the beam, and the actual shear diagram is something like that represented by the broken curved lines.

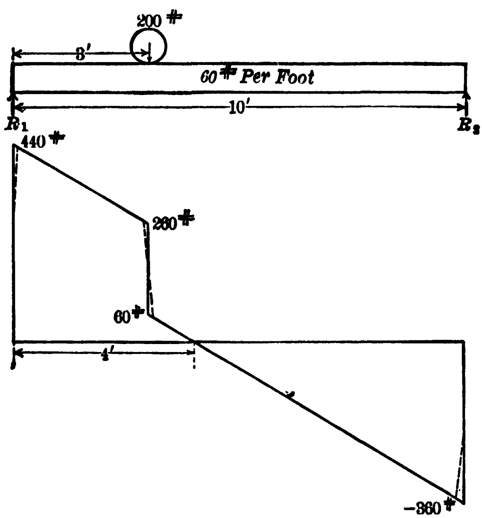


FIG. 102. Concentrated and distributed loads.

In Fig. 102, the shear decreases 60 pounds for every foot. With the origin of coordinates at the left end, the equation of shear for the first 3 feet is

$$V_0^3 = 440 - 60x$$

For the remainder of the beam,

$$V_3^{10} = 440 - 200 - 60x = 240 - 60x$$

Problems

57-4. Draw the shear diagram for the beam of Prob. 56-1. First draw the beam to a scale of 1 in. = 4 ft. Directly below and to the same horizontal scale draw the shear diagram using 1 in. = 1,000 lb. Write the shear equations, putting on limits. Find where the shear is zero.

57-5. Draw the shear diagram for the beam of Prob. 56-2. Draw the beam to a scale of 1 in. = 4 ft at the top of the paper. Directly below and to the same horizontal scale draw the shear diagram using 1 in. = 1,000 lb. Write the shear equations with limits. Find where the shear is zero.

57-6. Draw the shear diagram for the beam of Prob. 56-3. Draw the beam first using a scale of 1 in. = 4 ft, and directly below draw the shear diagram with a scale of 1 in. = 1,000 lb. Write the shear equations, putting limits on them. Find where the shear is zero.

57-7. A beam 16 ft long, supported at 4 ft from the left end and 2 ft from the right end, carries 120 lb on the left end and 240 lb on the right end. Construct the shear diagram to the scale of 1 in. equals 4 ft of length and 1 in. equals a shear of 100 lb.

57-8. Draw a sketch of the beam with its loads for each of the shear diagrams in Fig. 103.

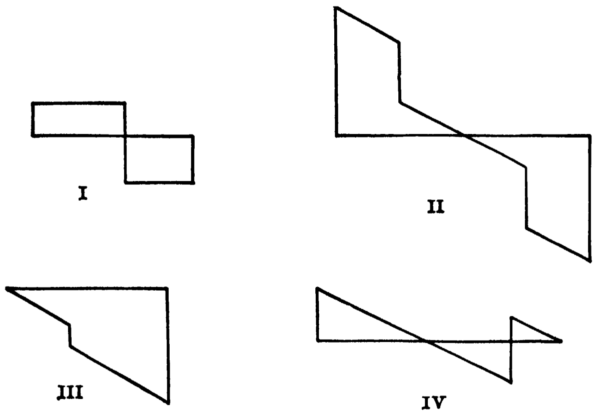


FIG. 103. Shear diagrams.

58. Bending Moment and Resisting Moment. In Fig. 98, the negative vertical shear is resisted by the tension in the vertical chain. In a horizontal beam subjected to vertical forces, the vertical shearing stress of the material resists the external vertical shear. In so far as vertical linear displacement is concerned, these forces produce equilibrium for the portion of the beam that is regarded as the free body. However, if the cylinder of Fig. 98 should be removed or crushed, the free portion of the beam would *rotate* around the intersection of the chains. The vertical loads and reaction produce a moment with respect to any horizontal axis in the section. For equilibrium, this *external* moment must be balanced by the *resisting* moment at the section. If moments are calculated with respect to the line of contact of the cylinder of Fig. 98 with the free portion of the beam, the moment of the vertical chain is zero. The force P causes a counterclockwise moment $P \times (x - a)$ and the distributed load causes a counter-

clockwise moment $wx \times (x/2)$. The clockwise moment of the horizontal tension H must balance the resultant moment of these loads. Expressed as couples, the resultant of P and wx downward forms a counterclockwise couple with the equal, upward force V exerted by the vertical chain. This couple is balanced by the clockwise couple formed by the equal, opposite horizontal forces H and C. Since these are couples, the moment of either is the same with respect to any axis perpendicular to their common vertical plane. It is convenient to calculate the moment about any axis in the plane of the section, provided this axis is perpendicular to the plane of the external forces.

Since the resisting moment and the external moment are equal, it is customary to speak of the *moment* at the section.

To calculate the *bending moment* at any section of a beam, multiply each force to the left of the section by its lever arm. Upward forces and reactions cause positive moment, and downward forces are negative. It is also correct to calculate the bending moment by using all the forces to the right of the section. The sign of the moment will be correct also, by using the positive sign on upward forces and the negative on downward forces. *Only those forces on one side of the section are to be used.*

When the curvature of the beam is convex upward (like the brim of a rain hat, curved so that the water runs off) the moment is negative. When the curvature is concave upward (so that it catches water like a trough) the moment is positive.

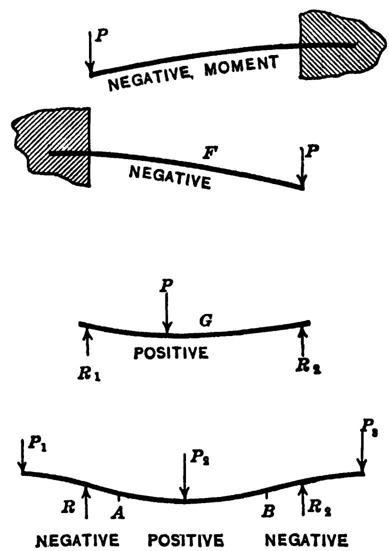


FIG. 104. Positive and negative moment.

The moment at any section of a horizontal beam is the same in magnitude and sign, whether calculated from the left or the right portion or viewed from the front or the rear. The shear calculated from the left end has the *same magnitude* as the shear calculated from the right end, but the *sign* is *opposite*.

The sections at A and B in Fig. 104, at which the moment changes sign, are called *points of inflection* or *points of contraflexure*. Bending moment is represented by M in formulas and equations.

Example 1

A beam 16 ft long is supported at the right end and 4 ft from the left end. It carries a uniformly distributed load of 60 lb per ft. A load of 240 lb is on the left end and a load of 360 lb is 6 ft from the right end. Find the moment and shear caused by these loads at intervals of 4 ft.

First make a sketch with dimensions and loads:

$$\text{Total distributed load} = W = wl = 60 \times 16 = 960 \text{ lb}$$

For external reactions, this load may be regarded as concentrated at the middle of the beam. Its position may be represented by a cross on the sketch. If this force is represented by an arrow, there is danger of using it incorrectly as a concentrated load in the calculation of internal moment or shear.

Taking moments about the right support:

$$R_1 = 1{,}140 \text{ lb}$$

Taking moments around the left support:

$$R_2 = 420 \qquad 420 + 1{,}140 = 1{,}560, \text{ check.}$$

After the reactions have been calculated and checked, they should be written on the sketch. Calculated numbers that are not part of the original data should be

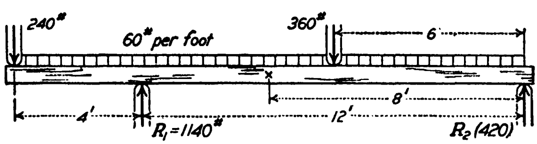


FIG. 105. Distributed and concentrated loads.

designated by parentheses or preceded by an equality sign. Figure 105 shows the sketch as it appears after the reactions are written.

At 4 ft from the left end:

$$V_{4-} = -240 - 4(60) = -480 \qquad V_{4+} = -240 - 240 + 1{,}140 = +660 \text{ lb}$$
$$M_4 = -240(4) - 60(4)(2) = -1{,}440 \text{ ft-lb}$$

At 8 ft from the left end:

$$V_8 = 420 \qquad M_8 = 720$$

At 12 ft from the left end:

$$V_{12} = -180 \qquad M_{12} = 1{,}200$$

At 16 ft from the left end:

$$V_{16-} = -420 \qquad M_{16} = 0$$

The expression V_{4-} means the shear infinitely close to the left of the reaction at 4 ft. The shear infinitely close to the right of this reaction is designated by V_{4+}. This shear is $-480 + 1{,}140 = 660$.

Problems

58-1. In the example above, find the shear and moment at 6, 9, 10, and 14 ft from the left end.
Ans. $V_6 = 540$, $M_6 = -240$; $V_9 = 360$, $M_9 = 1{,}110$; $V_{10-} = 300$, $V_{10+} = -60$, $M_{10} = 1{,}440$.

58-2. A beam of length l is simply supported at the ends and carries a load P at the middle. Find the moment at the middle, at one-third the length from the left end, and at two-thirds the length from the left end.
Ans. At the middle, $M = Pl/4$.

58-3. A beam of length l is simply supported at the ends and carries a load P at the middle. Find the expression for the moment at a distance x from the left end when x is not greater than one-half the length. Find the expression for the moment when x is greater than one-half the length.
Ans. $M = \dfrac{Px}{2}$; $M = \dfrac{Px}{2} - P\left(x - \dfrac{l}{2}\right) = \dfrac{P(l - x)}{2}$.

58-4. A simply supported beam of length l carries a uniformly distributed load of w per unit length. Find the moment at the middle, at one-third the length from the left end, and at two-thirds the length from the left end.
Ans. Moment at the middle $= wl^2/8$.

58-5. In Prob. 58-4, find the expression for the moment at a distance x from the left end. Also find the shear.
Ans. $M_x = \dfrac{wlx}{2} - \dfrac{wx^2}{2} = \dfrac{wx}{2}(l - x)$; $V_x = \dfrac{wl}{2} - wx$.

Moment diagrams are constructed in the same way as shear diagrams. The abscissas represent horizontal distances in the beam, and the ordinates represent the bending moments. In this book, positive moment is drawn upward.

Since shear diagrams usually consist of straight lines, they are easy to construct. Moment diagrams are curved, except when all the loads are concentrated.

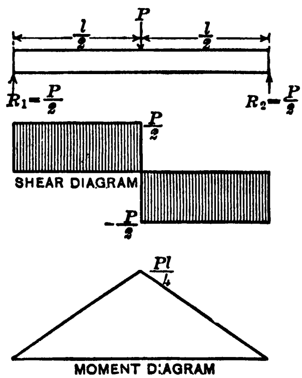


FIG. 106. Singly concentrated load.

Figure 106 shows the shear and moment diagram for a beam supported at the ends and carrying a load P at the middle. The weight of the beam is neglected. The end reactions are $P/2$. The moment at any section at a distance x from the left end is $Px/2$, provided x is not greater than one-half the length. Under the load the moment is $Pl/4$. The moment diagram for the left half of the beam is a straight line through the points (0,0) ($l/2$, $Pl/4$). Beyond the concentrated load, the moment caused by the reaction at the left end is

diminished by the moment caused by the load at the middle. At a distance x from the left end, when x is greater than $l/2$, the moment equation is

$$\text{Moment} = \frac{Px}{2} - P\left(x - \frac{l}{2}\right) = \frac{Pl}{2} - \frac{Px}{2} = \frac{P}{2}(l - x)$$

This also is a straight line. The last of the expressions for the moment may be obtained directly by using the portion to the right of the section as the free body. The right reaction is $P/2$ and its moment arm is $l - x$.

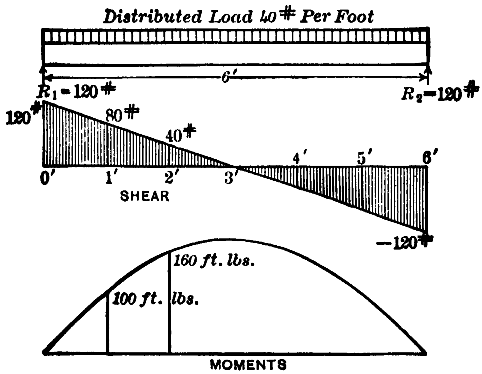


FIG. 107. Uniformly distributed load.

Figure 107 gives the shear and moment diagrams for a beam which is supported at the ends and carries a uniformly distributed load over the entire span. Always draw the beam, the shear diagram, and the moment diagram in that order, to the same horizontal scale. For the general case of a beam of length l with a load of w per unit length, the end reactions are $wl/2$. At a distance x from the left end, the moment of the end reaction is $wlx/2$. The uniformly distributed load over the length x is $-wx$. The moment arm of this distributed load with respect to the section at a distance x from the end is $x/2$, and the moment of the distributed load about this section is $-wx^2/2$. Moment at any section is given by

$$M = \frac{wlx}{2} - \frac{wx^2}{2} = \frac{wx}{2}(l - x)$$

Since x and $l - x$ appear in the first degree, it is evident that the curve is symmetrical with respect to a vertical line through the middle; the line $x = l/2$.

Example 2

A beam 12 ft long weighs 20 lb per ft. It is supported at the left end and 2 ft from the right end and carries a load of 100 lb 2 ft from the left end and a load of 80 lb at the right end. Construct the shear and moment diagrams to the scale of 1 in. horizontal equals 2 ft of length, and 1 in. vertical equals 100 lb shear and 100 ft-lb moment.

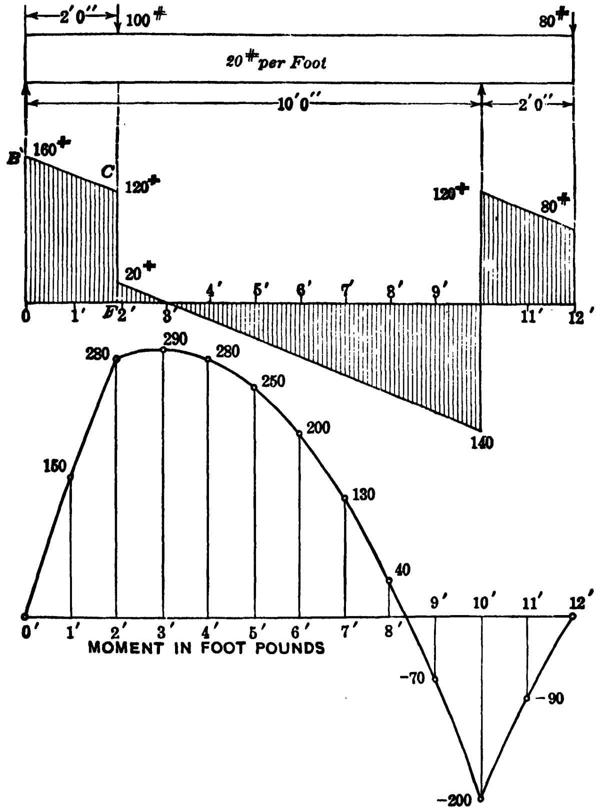


FIG. 108. Shear and moment diagrams.

Figure 108 shows the curves for this example.

Problems

(Construct all diagrams with shear directly below the sketch of the beam, and moment directly below shear.)

58-6. A simply supported beam is 10 ft long and carries a uniformly distributed load of 200 lb per ft over 6 ft adjacent to the left support, and no load over the remainder. Construct shear and moment diagrams to the scale of 1 in. equals 2 ft of length, 1 in. equals 500 lb of shear, and 1 in. equals 500 ft-lb.

Write the shear equations and the moment equations taking the origin at the left end. Calculate the shear at only three points. Calculate the moment at intervals of 1 ft for the first 6 ft.

58-7. A beam 12 ft long is supported at the right end and 2 ft from the left end. It carries a uniformly distributed load of 20 lb per ft, a load of 80 lb on the left end, and a load of 100 lb 2 ft from the right end. Construct the shear and moment diagrams to the scale of 1 in. equals 2 ft of length and 1 in. equals 100 lb of shear and 100 ft-lb of moment. Write the shear and moment equations, using the left end as origin.

58-8. In Prob. 58-7, change the 100-lb load to 120 lb and the uniform load to 50 lb per ft. Solve.

58-9. A uniformly loaded beam 14 ft long is supported at the left end and 4 ft from the right end. Draw the shear and moment diagrams.

59. Relation of Moment and Shear. If two vertical sections are taken through the beam of Fig. 108, the free body of Fig. 109 may be obtained. Since the body is in equilibrium under the action of the load and the internal forces, moments can be taken about some convenient point, such as the lower left corner,

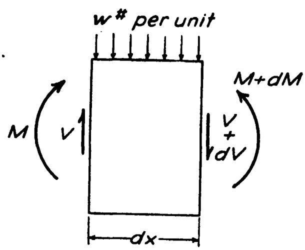


FIG. 109.

$$(M + dM) - M - (V + dV)\,dx - w\,dx\,\frac{dx}{2} = 0 \qquad (59.1)$$

Neglecting terms containing orders higher than the first and dividing by dx,

$$\frac{dM}{dx} = V \qquad \text{Formula XII}$$

The derivative, with respect to the length, of the moment equation of a beam gives the shear in the beam.

The above proof is based on the internal forces. The equation may be checked using the external forces of Fig. 108. Taking the origin at the left end of the beam, the moment equation for a portion of the beam is

$$M_2^{10} = 160x - 100(x - 2) - 20\frac{x^2}{2} \qquad (59.2)$$

Differentiating gives the shear equation

$$\frac{dM}{dx} = 160 - 100 - 20x = V_2^{10} \qquad (59.3)$$

The maximum moment may be found after solving Eq. (59.3) for x.

$$x = 3 \text{ ft}$$

This could have been determined by noting where the shear diagram crossed the axis. In fact it is better to determine the position of the maximum moment by drawing the shear diagram. The shear diagram indicates a zero shear also at the right reaction, but if the equation for moment is differentiated, this position will be missed.

In Fig. 108, there is an abrupt change in the slope of the moment curve at the concentrated load and at the second support. The shear at this point may be said to have any value between 120 and 20 pounds. The derivative of the moment is not *single valued* and Formula XII does not hold. It does hold, however, infinitely close to this point on either side.

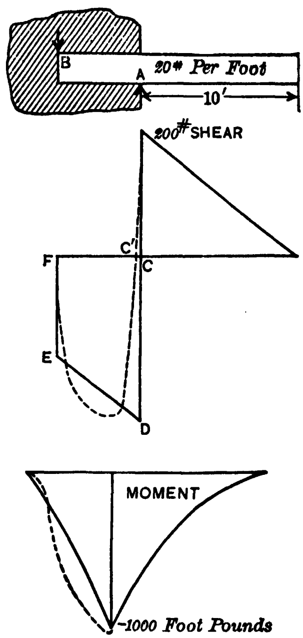


FIG. 110. Cantilever fixed at left end.

In reality, no load can be concentrated at a point or on a line extending across the beam. A so-called *concentrated load* is actually distributed over an area. If this distribution were known, the shear at any point would have a single value and Formula XII would be found to be valid at all sections.

A section in a beam where the moment has a maximum numerical value is called a *dangerous section*. The mathematical condition for a maximum or minimum value of M is that the derivative with respect to the length shall be zero. But since dM/dx is the shear, this means that there is a dangerous section at every point where the shear becomes zero.

The shear may pass through zero when the moment equation does not fulfill the mathematical condition that the slope of the tangent to the curve is zero. At the right support in Fig. 108, the slope of the moment curve changes abruptly from negative to positive. The negative moment at this point has the maximum numerical value. This is evident from the shear diagram.

The engineer uses the terms *maximum positive moment* and *maximum negative moment*. In Fig. 110 there is only one maximum moment. It is negative. Beams can have several maximum positive and negative moments. At all these points there is a dangerous section. Each maximum moment must be calculated, since the above theory does not tell "how big" each may be.

Figure 111 is part of the shear diagram for a beam which is supported at the left end, weighs w pounds per foot, and carries a load P at a distance a from the left end. The element of width dx at a distance x from the left of the diagram extends from the X axis to the line BC. The area of this element is $V\,dx$. The integral of $V\,dx$ between the

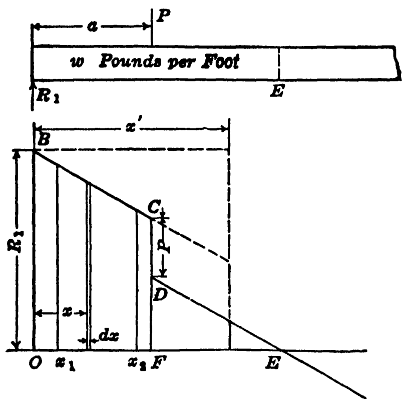


FIG. 111. Relation of area of shear diagram to moment.

limits x_1 and x_2 represents the area which is bounded by the shear diagram, the X axis, and the ordinates x_1 and x_2. Since

$$\int_{x_1}^{x_2} V\,dx = M_2 - M_1,$$

the area of the shear diagram between two points is the difference between the moments at these points. When the shear is negative, the area is below the X axis and, therefore, is negative.

Example

In Fig. 108, find the moment of 2, 3, 4, and 7 ft from the left end by means of the area of the shear diagram.

At 2 ft, the moment is the area of the trapezoid the base of which is 2 ft, and the altitude is 160 units on one side and 120 units on the other side.

$$M_2 = 0 + \frac{160 + 120}{2} \times 2 = 280 \text{ ft-lb}$$

At 3 ft, the moment is the moment at 2 ft plus the area of the triangle 20 units high and 1 ft wide.

$$M_3 = 280 + 10 = 290 \text{ ft-lb}$$

At 4 ft, the negative triangle is subtracted from the moment at 3 ft.

$$M_4 = 290 - 10 = 280 \text{ ft-lb}$$

At 7 ft, the negative triangle has a base of 4 ft and an altitude of 80 units

$$M_7 = 290 - \frac{80 \times 4}{2} = 130 \text{ ft-lb}$$

When the loading of a beam is given, always find the dangerous sections by means of a sketch of the shear diagram.

Problems

59-1. A simply supported beam of length l carries a uniformly distributed load over six-tenths of the length adjacent to the left support, and no load over the remainder. Find the moment at the dangerous section.
Ans. $M = 0.0882wl^2$.

59-2. A beam 12 ft long, weighing 60 lb per ft, is supported at the ends and carries a load of 240 lb 3 ft from the left end and a load of 720 lb 2 ft from the right end. Find the moment at the dangerous section.

59-3. Find the moment at the dangerous section for a cantilever 10 ft long, which carries a distributed load of 20 lb per ft, including its own weight.

59-4. A beam 20 ft long is supported 5 ft from the left end and 3 ft from the right end. It carries 360 lb on the left end and 600 lb on the right end. Draw the shear diagram. Draw the moment diagram. Locate the dangerous sections.
Ans. Dangerous section anywhere between supports.

59-5. A beam 20 ft long is supported at the ends and carries 320 lb 7 ft from the left end, and 560 lb 4 ft from the right end. Draw the shear diagram and the moment diagram. Find the dangerous section.

59-6. A simply supported beam 20 ft long carries a distributed load of 1,200 lb per ft, a load of 3,000 lb 4 ft from the left support, and a load of 6,000 lb 6 ft from the right support. Find the reactions and check. Draw the shear diagram. Find the dangerous section. Calculate the moment at the dangerous section by definition. Check by area of shear diagram. Find the moment 4 ft from the left end by shear trapezoid. Find the moment 6 ft from right end by area between that section and the dangerous section.

59-7. A beam 24 ft long is supported 4 ft from each end. It carries a distributed load of 500 lb per ft, including its own weight, and a load of 2,000 lb on the right end. Find the dangerous sections. Calculate the moment at the dangerous sections. *Ans.* One maximum moment is 8,250 ft-lb.

60. Miscellaneous Problems

60-1. A cantilever of length l is fixed at the right end and carries a load of w per unit length over six-tenths of the length adjacent to the free end. Draw the shear diagram. Calculate the moment at each two-tenths of the length from the definition of moment. From the end of load to the fixed end the moment diagram is a straight line. Draw the moment diagram.

60-2. A beam of length l is supported at the ends and carries a load w per unit length over six-tenths of the length adjacent to the left end. Draw the shear diagram. Find the moment at the dangerous section and at the middle.

60-3. A beam 16 ft long, weighing 120 lb per ft, is supported 3 ft from the left end and 1 ft from the right end. It carries 300 lb on the left end, 660 lb on the right end, and 1,080 lb 5 ft from the left end. Draw the shear diagram

and locate each dangerous section. Find the moment at each dangerous section algebraically and check by the area of the shear diagram. Write the equation of moments for the portion between the left support and the load of 1,080 lb and solve for the position of zero moment. The equation has two roots. Which one should be taken? Why? Write the equation and find the other position of zero moment.

Ans. One maximum moment is 2,220 ft-lb.

60-4. A beam 20 ft long is supported at 5 ft from the left end and at the right end. It carries a uniformly distributed load of 600 lb per ft including the weight of the beam, and also two concentrated loads: 3,000 lb on the left end, and 6,000 lb at 2 ft from the right end. Locate the dangerous sections and calculate the maximum moments.

60-5. In Prob. 60-4 write the shear and moment equations. Find the point of contraflexure.

60-6. A beam 20 ft long is supported at 4 ft from each end and carries 900 lb per ft uniformly distributed. There are two concentrated loads, 500 lb on each end. Find the maximum moment.

60-7. A beam is 16 ft long and is supported at 4 ft from the left end and 2 ft from the right end. It carries a uniform load of 100 lb per ft and three concentrated loads: 600 lb at 6 ft from the left end, 400 lb at 6 ft from the right end, and 400 lb on the right end. Draw the shear diagram and calculate the maximum moments.

60-8. Write the shear and moment equations for the beam of Prob. 60-7. Find the points of contraflexure.

CHAPTER 7

BENDING STRESSES IN BEAMS

61. Distribution of Stress. At any section of a bent beam, there is tension across the part adjacent to the convex surface and compression across the part adjacent to the concave surface, and there is usually shear parallel to the section. The method of finding the total vertical shear has been given in Chap. 6. The method of determining the unit shearing stress will be given later in Chap. 8. The problem of the total tension and compression and of the unit tensile and compressive stresses will now be considered.

If the external forces have no components parallel to the length of the beam, the resultant compressive stress across any section is equal to the resultant tensile stress, and these two forces form a couple, the moment of which is equal to the product of either force multiplied by the distance between them. This moment is equal and opposite to the bending moment.

To calculate these forces (H and C of Figs. 96 to 99), it is only necessary to know the bending moment and the distance between the forces. This distance is easily measured in Figs. 96 and 98.

In Fig. 99 the tensile stress is distributed over the entire upper portion and the compressive stress is distributed over the entire lower portion. In order to find the moment arm of the couple, it is necessary, therefore, to assume a law of distribution of these stresses.

The fibers on the convex side of a bent beam are elongated and those on the concave side are shortened. Between these there is a surface in which the fibers suffer no deformation in the direction of the length of the beam. This surface is called the *neutral surface* of the beam. The intersection of the neutral surface with any transverse section of the beam is called the *neutral axis* of that section.

It is customary to assume that the unit stress at any section varies directly as the distance from the neutral axis. Figure 112 represents graphically the variation of unit stress in a beam, the upper part of which is in tension.

Figure 112,I shows the forces from left to right and also the forces from right to left. Figure 112,II shows only the forces with which

the portion of the beam to the right of the section acts on the portion to the left. It will be noticed that both sets of forces tend to turn the left portion clockwise about the neutral axis at O. Figure 112,III

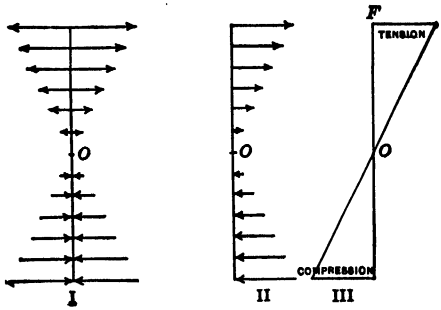


FIG. 112. Stress variation in a beam.

shows a convenient method of drawing the diagram to show the magnitude of the unit stress at any distance from the neutral axis.

Since the unit stress varies as the distance from the neutral axis, it may be represented by two wedges cut from the beam by two planes which pass through the neutral axis. One of these planes should be normal to the length of the beam and, therefore, represent the section considered, and the other may make any convenient angle. Figure 112,III may be considered as representing two such planes. The volume of each wedge may be regarded as giving the total stress across its corresponding part of the section, and the distance between the center of gravity of the two wedges, measured parallel to the section, gives the moment arm of these total stresses.

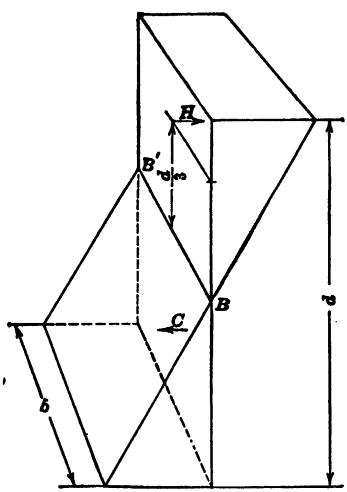


FIG. 113. Solids representing stress in rectangular prism.

62. Fiber Stress in a Beam of Rectangular Section. Figure 113 shows the wedges representing the stress distribution of a rectangular beam section of breadth b and depth d. Since the total tension H is equal to the total compression C, the two wedges which represent the total tension and compression must have equal volume. Since the slope and width of the wedges are the same and their volumes are equal, their heights must be equal, and the

neutral axis BB' is at a distance $d/2$ from the top or bottom of the section. If S is the unit tensile stress in the outer fibers at the top of the beam, $S/2$ is the average tensile stress over the upper half of the section. The total tension is the average stress multiplied by the area above the neutral axis.

$$H = \frac{S}{2} \times \frac{bd}{2} = \frac{Sbd}{4} \tag{62.1}$$

Problems

62-1. A beam of rectangular section is 4 in. wide and 12 in. deep. The unit stress in the outer fibers at the convex surface is 1,000 psi. What is the total tension? *Ans.* $500 \times 4 \times 6 = 12{,}000$ lb.

62-2. A beam of rectangular section is 6 in. wide and 10 in. high. The total tension is 4,800 lb. What is the average tensile stress? What is the maximum tensile stress? *Ans.* Maximum stress = 320 psi.

62-3. Figure 114 represents a T section with sides parallel. The neutral axis is 3 in. from the top of the flange. The unit stress at the top is given as

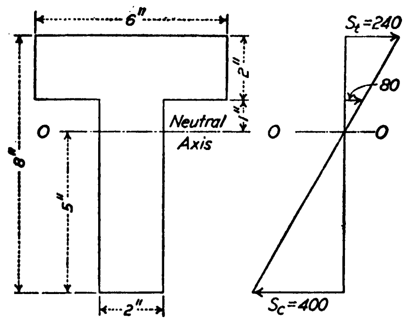


FIG. 114. Determination of neutral axis.

240 psi. What is the unit tensile stress at the bottom of the flange? What is the unit compressive stress at the bottom of the stem? What is the unit compressive stress 1 in. from the bottom?

Ans. $s_t = 80$ psi; $S_c = 400$ psi; $s_c = 320$ psi.

62-4. Calculate the total tensile stress in the beam of Prob. 62-3. Calculate the total compressive stress in the lower 5 in. of the stem.

		Lb
Ans. Tension in flange....................	$\frac{240 + 80}{2} \times 12 =$	1,920
Tension in upper inch of stem.......	$\frac{80 + 0}{2} \times 2 =$	80
Total tension	$=$	2,000
Compression in stem...............	$\frac{400 + 0}{2} \times 10 =$	2,000

62-5. The flange of a T beam is 5 in. wide and 2 in. high. The thickness of the stem is 1 in. and the net height is 10 in. The neutral axis is 4 in. from the

top of the flange or 8 in. from the bottom of the stem. The tensile stress at the bottom of the stem is 1,200 psi. What is the compressive stress at the top of the flange? What is the average compressive stress in the flange? What is the total tension in the lower 8 in. of the stem? What is the total compression in the remainder of the section?

Ans. Total tension = 4,800 lb.

62-6. A hollow box girder is 10 in. wide and 12 in. high outside and is 6 in. wide and 8 in. high inside. The maximum unit stress in the top and bottom fibers is 1,200 psi. Find the total tension in one half.

HINT: Subtract the inner part from the solid. *Ans.* 26,400 lb.

The line of application of the resultant tension H of Fig. 113 passes through the center of gravity of the wedge. Since the center of gravity of a triangle and, consequently, of a triangular wedge is two-thirds the height from the vertex, the distance of H from the neutral axis is $\frac{2}{3} \times \left(\frac{d}{2}\right) = \frac{d}{3}$. In like manner the total compression C is located at a distance $d/3$ below the neutral axis. The total moment arm of the couple made up of the forces H and C is $2d/3$.

Problems

62-7. What is the moment about the neutral axis of the forces of Prob. 62-1?

Ans. 12,000 × 4 + 12,000 × 4 = 96,000 in.-lb.

62-8. What is the moment about the neutral axis of the tensile stress of Fig. 114? Construct a three-dimensional sketch of the stress distribution solid.

The middle 2 in. of the stress distribution solid is a triangular wedge. The remainder makes two trapezoidal wedges. Regard the whole tension part as a triangular wedge 6 in. wide and 3 in. high. Subtract from the moment of this wedge the moment of two triangular wedges each 2 in. wide and 1 in. high.

Total tension of triangular wedge 6 in. wide and 3 in. high is

$$\frac{240 + 0}{2} \times 18 = 2{,}160 \text{ lb}$$

Total tension of two triangular wedges each 2 in. wide and 1 in. high is

$$\frac{80 + 0}{2} \times 4 = 160 \text{ lb}$$

Ans. Moment = 2,160 × 2 − 160 × $\frac{2}{3}$ = 4,320 − 106.7 = 4,213.3 in.-lb.

62-9. Find the moment about the neutral axis of the compressive stress of Prob. 62-3. Find the total moment. *Ans.* 6,666.7 in.-lb; 10,880 in.-lb.

62-10. Find the total moment about the neutral axis of the forces of Prob. 62-6.

Ans. (36,000 × 4 − 9,600 × $\frac{8}{3}$)2 = 236,800 in.-lb.

62-11. Find the total moment about the axis of the forces of Prob. 62-5.

Ans. M = 25,600 + 14,400 = 40,000 in.-lb.

The total resisting moment of a rectangular section is

$$M = \frac{Sbd}{4} \times \frac{2d}{3} = \frac{Sbd^2}{6} \tag{62.2}$$

This fundamental equation gives the relation between the bending moment and the maximum bending stress in a rectangular beam.

In problems of rectangular beams the horizontal dimension will be given first, when discussing the usual horizontal beam subjected to vertical loads.

Example

A 4- by 6-in. cantilever carries a load of 240 lb on the free end. Find the unit stress in the top and bottom fibers at a section 5 ft from the free end.

Horizontal dimensions are given first. A 4- by 6-in. beam is 4 in. wide and 6 in. deep. Since unit stresses are required in pounds per square inch, the moment must be in inch-pounds.

$$M = \frac{Sbd^2}{6}$$

$$S = \frac{6M}{bd^2} = \frac{6 \times 14{,}400}{4 \times 36} = 600 \text{ psi}$$

Problems

62-12. A 6- by 10-in. beam 12 ft long is supported at the ends and carries a load of 2,400 lb at the middle. Calculate the unit bending stress in the top and bottom fibers at the dangerous section caused by this load. *Ans.* $S_t = S_c = 864$ psi.

62-13. A 10- by 6-in. beam 12 ft long is supported at the ends and carries a 2,400-lb concentrated load at 4 ft from the left end. Find the maximum bending stress and indicate on your sketch where it occurs.

62-14. An 8- by 12-in. beam is 15 ft long and is supported at the ends. It carries 1,000 lb 6 ft from the left support and a distributed load of 240 lb per ft. Find the maximum unit bending stress. *Ans.* $S = 630$ psi.

62-15. An 8- by 12-in. beam is 15 ft long and is supported at the ends. It carries 600 lb per ft and 1,000 lb at 6 ft from the left end. Find the maximum bending stress. *Ans.* 1,250 psi.

62-16. A 4- by 6-in. beam 8 ft 4 in. long, is supported at the ends and carries a load at the middle which makes the maximum stress 900 psi. Find the load. *Ans.* $P = 864$ lb.

62-17. The beam of Prob. 62-16 has the 6-in. faces horizontal. What is the stress for a load of 864 lb at the middle?

63. Fiber Stress in a Beam of Any Section. The methods of Art. 62 are not convenient for sections which are not rectangles. There is a general method, however, which applies to any form of section.[1]

Figure 115 may be regarded as representing a section of any form.

[1] It was Navier (1785–1836), a French professor and engineer, who brought the flexure formula to its modern state of development.

BB' is the neutral axis. An element of area dA is at a distance v from the neutral axis. The area dA may be infinitesimal in two dimensions or it may extend entirely across the section parallel to the neutral axis as shown by the broken lines.

Since the unit stress varies as v, it may be represented by kv, in which k is the unit stress at unit distance from the neutral axis. The total stress on the element dA is the unit stress times the area.

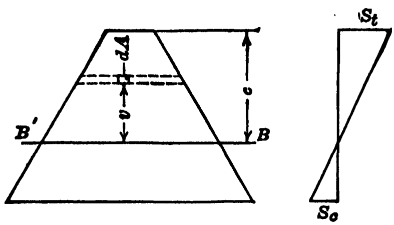


FIG. 115. Beam section.

$$\text{Total stress} = kv\,dA \tag{63.1}$$

The moment of this stress on dA about the neutral axis is

$$dM = kv^2\,dA \tag{63.2}$$

Since v^2 is positive when v is positive or negative, the sign of increment of moment is the same whether the element is above or below the neutral axis.

$$M = k\int v^2\,dA = kI \tag{63.3}$$

in which I is the moment of inertia of the section with respect to the neutral axis. Since

$$s = kv \qquad k = \frac{s}{v} \tag{63.4}$$

which, substituted, gives

$$M = \frac{sI}{v} \tag{63.5}$$

The most important stress is the stress in the extreme outer fibers where v is a maximum and the unit stress is the greatest. If the maximum unit stress is represented by S and the distance to the outer fiber from the neutral axis is represented by c, the equation becomes

$$S = \frac{Mc}{I} \qquad \text{Formula XIII}$$

The values of I and c in Formula XIII depend upon the location of the neutral axis. This is found from the condition that the total tensile stress across the part of the section on one side of the neutral axis is equal to the total compressive stress across the part of the section on the other side of the axis.[1] On an element dA,

[1] Coulomb was one of the early investigators to point out the correct location of the neutral surface.

$$\text{Total stress} = kv\,dA \tag{63.6}$$

$$\text{Total stress on entire section} = k\int v\,dA = 0 \tag{63.7}$$

The constant k is not zero when the beam is bent; consequently, $\int v\,dA$ must be zero.

The center of gravity of a plane area is given by

$$\bar{v} = \frac{\int v\,dA}{A} \tag{63.8}$$

$$\bar{v}A = \int v\,dA = 0 \tag{63.9}$$

Since A is not zero,

$$\bar{v} = 0 \tag{63.10}$$

The neutral axis of a beam of any section passes through the center of gravity of the section.

64. Section Modulus. The expression I/c, in which c is the distance from the neutral axis to the extreme outer fiber, is called the *section modulus* or *modulus of the section.* If the section modulus is represented by Z, Formula XIII becomes

$$S = \text{unit stress in outer fibers} = \frac{M}{Z} \tag{64.1}$$

The section moduli for rolled shapes and for the principal geometric figures are given in the handbooks of the steel and aluminum manufacturers.[1]

Most modern American textbooks represent stress by S and s. Several use the combination I/c for the section modulus.

For a rectangular section, $I = bd^3/12$ and $c = d/2$.

$$Z = \frac{I}{c} = \frac{bd^2}{6} \tag{64.2}$$

When this value of Z is substituted in Formula XIII,

$$M = \frac{Sbd^2}{6} \tag{64.3}$$

which shows that the rectangular beam is merely a special case.

[1] For information, see "Steel Construction Manual" of the American Institute of Steel Construction, and Alcoa Structural Handbook of the Aluminum Company of America. Much essential information is contained in the pamphlet, "Hot Rolled Carbon Steel Structural Shapes" of the United States Steel Corporation. All handbooks use S for section modulus and f for unit stress.

Always use the section modulus as given in the handbook or $bd^2/6$ for a rectangular section, instead of I/c. Beams of unsymmetrical section, such as T beams with stems vertical or channels with loads perpendicular to the web, have two different values of c and two corresponding values of section modulus. The handbook gives the modulus for the larger c and the largest S in Formula XIII. It is necessary, sometimes, to compute Z for the smaller c.

Problems

64-1. A 15-in. 42.9-lb standard I beam, 20 ft long between centers of supports, carries a uniformly distributed load of 1,200 lb per ft and a load of 6,000 lb 6 ft from the left support. Find the maximum fiber stress caused by these loads. *Ans.* S = 16,170 psi.

64-2. Find the unit stress in the outer fibers of the beam of Prob. 64-1 under the load and at the middle. What is the stress at the dangerous section 1 in. from the top? *Ans.* 15,400 psi; 15,890 psi; 14,014 psi.

64-3. An 8-in. 23-lb standard steel I beam 20 ft long is supported at the right end and at 5 ft from the left end. It carries 600 lb per ft including the weight of the beam and two concentrated loads: 5,000 lb at 3 ft from the right end and 1,500 lb on the left end. Find the maximum bending stress. *Ans.* 14,410 psi.

64-4. A 12-in. 11.31-lb standard aluminum I beam 18 ft long is supported at 4 ft from each end. It carries 1,500 lb per ft, which includes the weight of the beam, and two concentrated loads: 8,000 lb at 6 ft from the left end and 4,000 lb on the right end. Find the maximum bending stress in the beam. *Ans.* 9,240 psi.

64-5. A 5-in. 3.53-lb standard aluminum I beam 14 ft long is supported at 4 ft from the left end and at the right end. It carries 400 lb per ft including its own weight and a 600-lb concentrated load at 2 ft from the right end. Find the maximum bending stress. *Ans.* 10,390 psi.

65. Allowable Bending Stress. Table 14 gives some allowable bending stresses which are *to be memorized.* Some of these are from official specifications. Since the allowable stresses for materials will vary somewhat depending on the particular specification governing, the working stresses given are to be considered averages. They are to be used in problems where the particular stresses are not otherwise designated.

TABLE 14. BENDING STRESSES IN EXTREME FIBERS

Material	*Psi*
Rolled structural steel	20,000
Structural steel pins	25,000
Cast steel	16,000
Cast iron, tension in extreme fiber	3,000
Cast iron, compression in extreme fiber	16,000
Timber, good structural grades	1,200
Aluminum alloy 17S-T and 24S-T	15,000

Problems

65-1. A 15-in. 50-lb standard steel I beam is simply supported on a 25-ft span. The beam has a 12- by 1-in. plate welded to the bottom flange. Find the location of the neutral axis of the beam. Calculate the moment of inertia and section modulus of the combination. Find the total uniformly distributed load it may carry. *Ans.* 43,370 lb.

65-2. A beam is supported at the ends on a 15-ft span. It carries a uniform load of 3,000 lb per ft which includes the beam, and a 15,000-lb load concentrated at 6 ft from the left end. Find the maximum bending moment. If these loads are supported by a standard steel I beam under specifications which limit the maximum bending stress to 18,000 psi, select the best I beam.
Ans. An 18-in. 70-lb standard I beam would do, but a 20-in. 65.4-lb beam would be better and cheaper.

65-3. The loads of Prob. 65-2 are carried by a wide-flange (WF) section. Select the beam, using 18,000-psi allowable stress.
Ans. A 14-in. wide-flange 61-lb beam would do, but an 18-in. wide-flange 55-lb beam is better, assuming sufficient headroom.

65-4. Timber joists are 2 in. wide, d in. deep, and supported on a 12-ft span. Find the depth to carry a uniform load of 100 lb per ft.

65-5. A hollow box beam is made of four pieces of 2- by 8-in. timbers put together so that the outside dimensions are 8 by 12 in. What is the maximum safe span for a load of 300 lb per ft?

65-6. A rolled steel T section is 5 by 3 by $\frac{3}{8}$ in. It is supported on a 5-ft span with the stem down. Find the maximum safe uniformly distributed load.

65-7. Solve Prob. 65-6 if the beam is made of cast iron.

65-8. Solve Prob. 65-7 if the beam is made of cast iron and is used with the stem up and flange down.

65-9. A trapezoidal cast-iron beam is 3 in. wide at the bottom, 1 in. wide at the top, and 4 in. high. It spans 5 ft and carries a load 2 ft from one end. Find the total safe load.

65-10. A beam 20 ft long is supported at the left end and at 5 ft from the right end. There is a 10,800-lb load concentrated on the right end, and a uniform load of 3,000 lb per ft distributed over the 7 ft adjacent to the left end, with no uniform load on the rest of the beam. Neglect the weight of the beam itself and select a wide-flange steel beam to carry these loads under specifications which limit the allowable bending stress to 18,000 psi.

65-11. A beam 16 ft long is supported at 4 ft from the left end and 2 ft from the right end. It carries a 3,000-lb load concentrated on the right end, a 2,000-lb load concentrated at 6 ft from the left end, and 500 lb per ft uniformly distributed over the beam. Find the size of a square wood beam.

66. Restrictions Underlying Stress Theory. In the discussion of beams just presented, several assumptions were tacitly made. Some of these are so important to a clear understanding of beams that it is advisable to emphasize their implications. The assumptions necessary to the flexure formula are

1. Any plane transverse section before bending remains plane after bending.

2. The stresses are below the elastic limit and vary directly with the distance from the neutral axis.

3. The material is homogeneous and isotropic, *i.e.*, it has the same properties throughout.

4. The forces are applied perpendicular to one principal axis which passes through the center of gravity of the section.

5. The forces do not cause twisting, buckling, or crippling.

6. The longitudinal axis of the beam is straight.

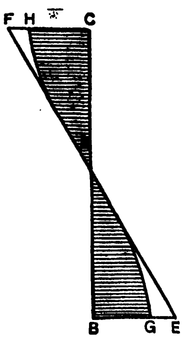


FIG. 116. Stress-distribution diagram beyond the elastic limit.

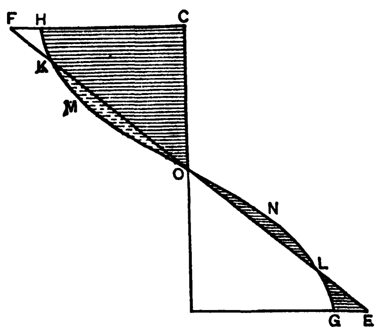


FIG. 117. Actual and calculated unit stress.

7. There are no abrupt changes of cross section where stresses are computed.

8. The beam is relatively narrow so that transverse lateral contractions or expansions (calculated by applying Poisson's ratio to the longitudinal strains) can be neglected.

67. Modulus of Rupture. In order to understand and interpret correctly the results of tests of beams loaded to failure, it is necessary to consider the effect of stresses beyond the elastic limit. The other seven of the eight assumptions given above will continue to be valid. Only No. 2 will be violated.

The straight lines in Fig. 116 represent the stress distribution diagram for a beam, no part of which has passed the elastic limit. The curved line *GH* would represent the stress distribution for a similar beam having the same modulus of elasticity but having a lower elastic limit. The resisting moment for the shaded areas is smaller than for the beam, which is still below the elastic limit.

In Fig. 117, let *HMONG* represent the stress distribution in a beam

just as failure occurs. If the exact shape of the curve were known, the bending moment could be equated to the resisting moment which is expressed in terms of *HC*, the maximum fiber stress. A solution gives the maximum bending stress at failure. It is customary, however, to assume that the stress distribution is still a straight line and Formula XIII is valid. The resisting moment is therefore a function of the moment of the area of *FOC*. The result obtained for S in Formula XIII at failure of the beam is called the *modulus of rupture.*

$$S_r = \frac{M}{Z} \qquad \text{Formula XIV}$$

While the modulus of rupture does not give the actual unit stress in the outer fibers, it makes it possible to compare stresses in similar *sections.* If the modulus of rupture of a given material is obtained from tests of beams of *rectangular* section, this modulus may be used in computing the ultimate strength of beams of this material of *any rectangular* section. The results may also be used with little error for beams of other shapes, provided they are symmetrical with respect to the neutral axis. With unsymmetrical sections, such as angles, it is better to make tests and obtain the modulus of rupture for each shape.

The student will remember, however, that these statements apply to the stress beyond the elastic limit. The change in the stress distribution diagram when the stress passes the elastic limit *affects the factor of safety only.*

Strictly speaking, ductile materials, such as soft steel, have no modulus of rupture, since beams of such material may be bent double without breaking.

The neutral axis may be shifted in a *symmetrical section,* if the tension and compression curves are not alike. In cast iron, for instance, the stress-strain diagram for tension differs greatly from the diagram for compression. The compressive strength of cast iron is three or four times as great as the tensile strength. Beams of this material should be made of T section, or equivalent, and so loaded as to bring the stem in compression and the flange in tension. The remote fibers on the compression side should be *two* or *three* times as far from the center of gravity of the section as those on the tension side.

Problems

67-1. A rectangular beam of western spruce, tested at the Bureau of Standards, was 1.75 in. wide and 1.78 in. high. The beam was on supports 24 in. apart

and loaded at the middle. When the total load changed from 31 to 597 lb, the deflection at the middle increased 0.1326 in. The beam failed at a load of 973 lb. Find the unit stress at 597-lb load, and find the modulus of rupture. *Ans.* 3,879 psi; 6,304 psi.

67-2. A rectangular bar of cast iron, 1.04 in. wide and 0.80 in. thick, was placed on supports 12 in. apart and was broken by a load of 1,635 lb midway between the supports. Find the modulus of rupture.

67-3. A cast-iron T beam was tested in bending with a concentrated load applied at the middle of a 20-in. span. The elements of the section are width of flange, 2 in.; stem depth (total), 1½ in.; thickness of flange and stem, ¼ to 5⁄16 in.; moment of inertia, axis 1–1, 0.16 in.^{4}; distance from axis 1–1 to back of flange, 0.42 in.; area, 0.91 sq in. When this beam was placed with flange down and loaded with 2,400 lb at the middle, it did not fail. What was the calculated stress at the bottom of the flange and at the top of the stem at this load? *Ans.* S_t = 31,500 psi; S_c = 81,000 psi.

67-4. When the T beam of Prob. 67-3 was turned over to bring the stem to the bottom, it failed under a load of 1,510 lb. Find the modulus of rupture. Find the unit compressive stress in the flange.
Ans. Tension modulus of rupture = 50,960 psi; S_c = 19,820 psi.

67-5. One-half of the beam of Prob. 67-4 was placed flange down across a 10-in. span. The load at failure was 7,610 lb. Find the modulus of rupture. Find the unit tensile stress in the flange.
Ans. Compression modulus of rupture = 128,400 psi; S_t = 49,940 psi.

67-6. A stoneware rod, average diameter 0.949 in., was supported at points 4.2 in. apart and carried two equal loads, each 4.2 in. outside the adjacent support. It failed under a total load of 102 lb. Find the modulus of rupture. *Ans.* 2,553 psi.

67-7. A 6- by 16-in. pine beam was supported at the ends on a 12-ft span and loaded at the third points. The beam failed when a 24,000-lb load was placed at 4 ft from each end. Find the modulus of rupture. *Ans.* 4,500 psi.

67-8. A 6- by 12-in. pine beam 12 ft long was supported at the ends and loaded with equal loads at 4 ft from each end. The beam failed when each load was 20,400 lb. Find the modulus of rupture.

67-9. A 4- by 6-in. wood beam 10 ft long was tested to failure by applying a concentrated load at the middle of the span. Find the modulus of rupture if the maximum load was 3,840 lb.

67-10. A 6- by 4-in. beam 10 ft long was cut from the same piece as the one in Prob. 67-9. It is to be used flatwise to carry a uniformly distributed load on a 10-ft span. Find the load if the allowable bending stress is obtained by applying a factor of safety of 6 to the modulus of rupture.

68. Miscellaneous Problems

68-1. A 4- by 5- by ½-in. aluminum T beam weighing 5.56 lb per ft is supported at the ends on a 10-ft span. It carries a uniform load of 500 lb per ft over the 8 ft adjacent to the right end and no load on the remainder. (Neglect the weight of the beam.) Find the maximum tensile and compressive bending stresses and indicate exactly where they occur in the beam.
Ans. S_t = 22,000 psi.

68-2. A beam is 24 ft long and is supported at 4 ft from the left end and 5 ft from the right end. It carries two uniform loads: 750 lb per ft over the 10 ft adjacent to the left end, and 600 lb per ft over the 14 ft adjacent to the right end. Find the size of a square wood beam.

68-3. A beam 18 ft long is supported at 4 ft from each end. It carries 1,000 lb per ft over the left half of the beam and 1,500 lb per ft over the right half of its length. If the maximum allowable bending stress is 16,000 psi, select a standard steel I beam to be used with the web horizontal.

68-4. A beam 20 ft long is supported at the right end and at 5 ft from the left end. It carries 600 lb per ft including the weight of the supporting beam, and three concentrated loads: 3,000 lb on the left end, 9,000 lb at 8 ft from the left end, and 7,500 lb at 4 ft from the right end. If the allowable bending stress is 16,000 psi, select two standard steel channels welded back to back to support these loads.

68-5. A 10- by 5¾-in. wide-flange 21-lb steel I beam has the web reinforced by welding a 9- by ¼-in. plate to each side of the web. The beam is 14 ft long, supported at the right end and at 4 ft from the left end, and carries a concentrated 10,000-lb load on the left end and 6,000 lb per ft over the 8 ft adjacent to the right end. Find the maximum bending stress in the beam. *Ans.* 22,270 psi.

68-6. A 7-in. 15.3-lb standard I beam has 3- by 1-in. plates welded to the top and bottom flanges. The beam is 14 ft long and is supported at the right end and at 4 ft from the left end. It carries two uniform loads which include the weight of the beam: 1,000 lb per ft over the left 6 ft and 2,000 lb per ft over the right 8 ft. Find the maximum bending stress and indicate on your sketch where it occurs. *Ans.* 8,240 psi.

68-7. Figure 118 shows a 2-in. diameter horizontal beam to which a couple is applied, such that $P = 300$ lb and a is 1 ft. The couple is applied 4 ft from the left end and 2 ft from the right end. Find the maximum bending stress in the beam. *Ans.* 9,168 psi.

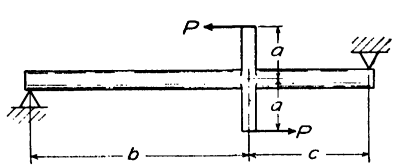


FIG. 118.

68-8. A 10-in. 25.4-lb standard steel I beam 21 ft long is supported at 6 ft from the left end and at the right end. Find the total uniformly distributed load it will carry.

CHAPTER 8

SHEARING STRESSES IN BEAMS

69. Direction of Shear. The total vertical shear in a beam is calculated by the methods of Art. 57, but this gives no information in regard to the distribution of the shearing stress in the section. In Art. 21 it was shown that shearing stresses occur in pairs, and that a small block subjected to shearing stress of given intensity along two parallel faces is subjected to a shearing stress of the same intensity along two other faces at right angles to these.

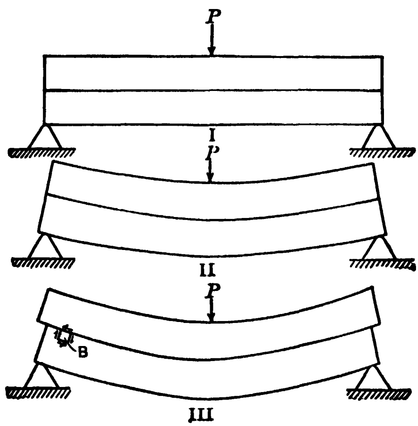


FIG. 119. Horizontal shear in beams.

Figure 119,I represents a beam made by placing one plank on top of another. Figure 119,II is the same beam under load, provided that the planks are held from slipping with reference to each other by being glued or bolted together to form a single beam. If the planks are free to move, they take the form III, in which the upper plank is moved outward over the lower one at each end. A small block B in the upper portion of the lower plank may be treated as a free body. The plank above this block has been displaced toward the left. If the planks were glued together, the upper plank would have exerted a horizontal shearing stress upon the upper surface of the block. To prevent rotation there must be a vertical shear upward at the left side. The actual

shearing stresses upon this block from the surrounding material, if the upper plank were glued to the lower, would take the directions of the arrows.

70. Intensity of Shearing Stress. Figure 120 represents a part of a beam subjected to vertical shear and to a bending moment. The shear is assumed to be positive from left to right and the moment is assumed to produce compression in the fibers above the neutral surface. A small block is shown extending across the beam between vertical planes dx apart and reaching from the top of the beam to a horizontal plane

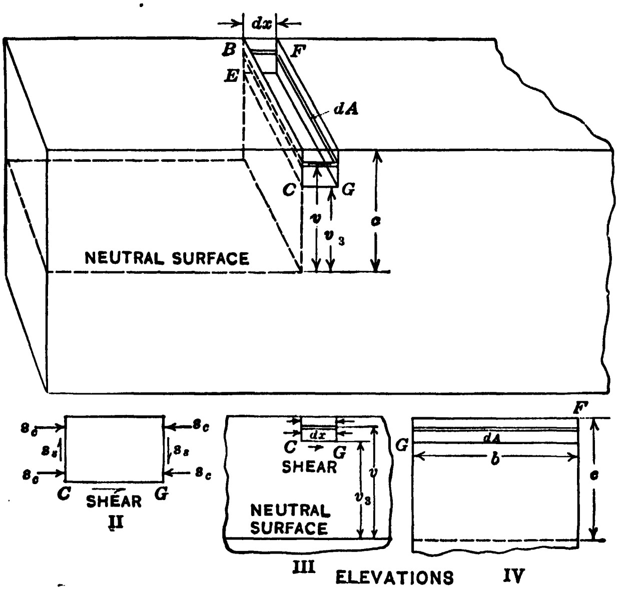


FIG. 120. Horizontal shear in rectangular section.

at a distance v_3 from the neutral surface. Two elevations of this block and the adjoining parts of the beam are shown in Fig. 120,III and IV, and an enlarged elevation of the block in Fig. 120,II. The block is in equilibrium under the action of the compressive stress on the ends (the rectangles whose diagonals are CB and GF), the vertical shearing stress on the same surfaces, the horizontal shear from the material below (on the rectangle GE), and the vertical compression or tension across the base.

In Fig. 120, a filament of cross section dA and length dx extends through the block parallel to the neutral surface. The unit compressive stress on the left end of this filament is M_1v/I_1, in which M_1 is the bending moment at the section, and I_1 is the moment of inertia with

respect to the neutral axis of the entire cross section of the beam. The compression on the left end of the filament is $\frac{M_1 v\,dA}{I_1}$. The total compression on the left end of the block is the integral from v_3 to c of the compression on the left end of the filament.

$$\text{Total compression on left end} = \frac{M_1}{I_1}\int_{v_3}^{c} v\,dA \tag{70.1}$$

$$\text{Total compression on right end} = \frac{M_2}{I_2}\int_{v_3}^{c} v\,dA \tag{70.2}$$

The resultant horizontal push on the block in the direction of the length of the beam is the difference of these integrals. If the section of the beam is uniform, $I_1 = I_2$, and v_3 and c are the same for both expressions. The resultant horizontal push on the block is

$$\frac{M_2 - M_1}{I}\int_{v_3}^{c} v\,dA \tag{70.3}$$

This resultant horizontal force must be balanced by the horizontal shear at the bottom of the block. If the breadth CE at the bottom of the block is b, the total area in horizontal shear is $b\,dx$, and the total shear is $s_s b\,dx$. When the resultant horizontal compression is equated to the horizontal shear at the bottom of the block,

$$s_s b\,dx = \frac{M_2 - M_1}{I}\int_{v_3}^{c} v\,dA \tag{70.4}$$

$$s_s = \frac{M_2 - M_1}{Ib\,dx}\int_{v_3}^{c} v\,dA \tag{70.5}$$

Since $M_2 - M_1$ is equal to dM,

$$\frac{M_2 - M_1}{dx} = \frac{dM}{dx} = V \tag{70.6}$$

in which V is the total vertical shear.

$$s_s = \frac{V}{Ib}\int_{v_3}^{c} v\,dA \tag{70.7}$$

in which s_s equals the unit horizontal shear at a distance v_3 from the neutral axis and also equals the unit vertical shear at the same place. The term $\int_{v_3}^{c} v\,dA$ is the moment of the area of the end of the block with respect to the neutral axis.

$$\bar{v} = \frac{\int_{v_1}^{c} v\,dA}{A} \qquad \int_{v_1}^{c} v\,dA = \bar{v}A \tag{70.8}$$

When the area and location of the center of gravity of the portion of the plane section above the line CE are known, the integral may be replaced by the equivalent expression.

$$s_s = \frac{V}{Ib}\bar{v}A' \tag{70.9}$$

in which V is the total vertical shear at the section, I is the moment of inertia of the *entire section* with respect to the neutral axis, b is the breadth of the section at the plane at which s_s is calculated, A' is the

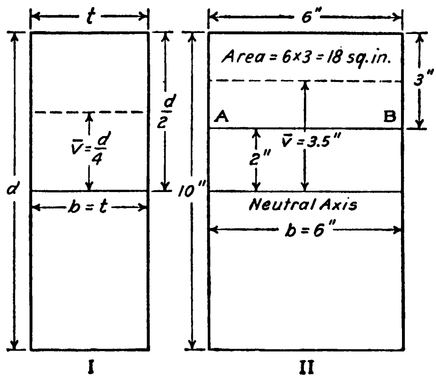


FIG. 121. Rectangular section in shear.

area of the *portion* of the section above (or below) this plane, and $\bar{v}$ is the distance of the center of gravity of A' from the neutral axis. The moment of the area, $\bar{v}A'$, is sometimes called the *statical moment* and designated by Q.

$$s_s = \frac{VQ}{Ib} \qquad \text{Formula XV}$$

Figure 121,I shows a rectangular section of thickness $t = b$. By inspection of the formula, the maximum shearing stress is at the neutral axis for a rectangular section. Substituting $I = bd^3/12$,

$$Q = \frac{d}{4} \times \frac{bd}{2}$$

$$S_s = \frac{3V}{2bd} = \frac{3}{2}\frac{V}{\text{area}} \tag{70.10}$$

Since the average vertical shear is V/total area, the unit horizontal (or vertical) shearing stress at the neutral surface of a rectangular section is three halves as great as the average unit vertical shearing stress.

Example

Find the unit shearing stress in a 6- by 10-in. rectangular section at a plane 3 in. from the top, when the total vertical shear is 6,000 lb (Fig. 121,II).

$$\frac{V}{Ib} = \frac{6{,}000}{500 \times 6} = 2 \qquad Q = 3.5 \times 18 = 63$$
$$s_s = 2 \times 63 = 126 \text{ psi}$$

In this solution, the unit shearing stress in the plane AB of Fig. 121,II has been found by means of the area above the plane. The area between AB and the bottom of the section might have been used.

Problems

70-1. In the example above, find the shearing stress on a horizontal plane 2 in. below the top of the beam. *Ans.* $s_s = 96$ psi.

70-2. In the example above, find the maximum shearing stress.

70-3. A 6- by 8-in. beam 8 ft long is supported at the ends and carries 6,400 lb at 5 ft from the left end. Find the maximum shearing stress and indicate on your sketch of the beam exactly where it occurs. Find the maximum bending stress and indicate where it occurs. *Ans.* $S_s = 125$ psi.

70-4. In Prob. 70-3, find the unit shearing stress at a point 1 ft from the left end and 1 in. below the top surface of the beam. *Ans.* 32.8 psi.

70-5. A 6- by 10-in. cantilever beam carries a 6,000-lb load on the end. Neglecting the weight of the beam itself, find the unit shearing stress at each inch from the top to bottom, and plot the stress as abscissa, using inches as ordinates. The diagram shows the variation of stress across a rectangular beam. *Ans.* $S_{s\max} = 150$ psi.

70-6. The beam in Prob. 70-5 is limited to a length so that the maximum bending stress does not exceed 1,200 psi. Find the maximum length.

70-7. A 7- by 16-in. beam of Douglas fir, supported at points 13 ft 6 in. apart and loaded at the third points with equal loads, failed by shear when the total load was 45,000 lb. Find the ultimate shearing strength of this timber parallel to the grain. *Ans.* 301 psi.

70-8. Timber having an allowable shearing stress of 90 psi and an allowable bending stress of 1,350 psi is used as a 6- by 8-in. beam to carry a load at the middle. What is the maximum allowable load on any length?
Ans. Maximum $V = 60 \times 48 = 2{,}880$ lb; maximum load at middle = 5,760 lb.

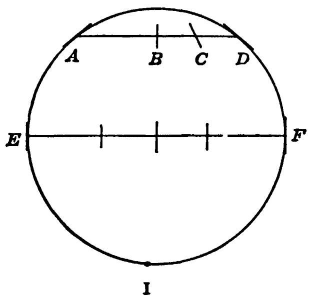
70-9. Below what length does shear govern in Prob. 70-8? On account of shear the load at the middle cannot exceed 5,760 lb.
Ans. $Z = 64$ in.^{3}; $M = 64 \times 1{,}350 = 86{,}400$; $2{,}880 \times l/2 = 86{,}400$; $l = 60$ in.

For a length greater than 60 in., the bending calculation gives a load less than 5,760 lb; bending governs. For a length smaller than 60 in. the

bending calculation gives a load greater than 5,760 lb. Since 5,760 lb is the maximum in shear, shear governs.

70-10. Solve Prob. 70-9 for a uniformly distributed load.

71. Nonrectangular Beams. Formula XV gives the unit shearing stress in the horizontal plane *GE* of Fig. 120. If the section of the beam is not rectangular, the unit shearing stress may not be uniform over the horizontal surface. Figure 122,I is a circular section in which *AD* is the trace of a horizontal plane. The short lines crossing *AD* are traces of planes in which the shear is transmitted from one side of *AD* to the other. At the middle of *AD* the shear is transmitted from a filament above the plane to a filament directly below it, and the line *B* is vertical. At *A* and *D*, the shear is transmitted from a filament on



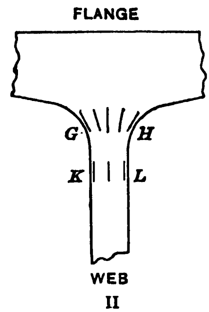


FIG. 122. Shear in curved sections.

the outer surface above the plane to a slightly larger filament, also on the outer surface, below the plane. The short lines at *A* and *D* are tangent to the section. At the diameter *EF*, the shear is transmitted from a filament above the plane to an equal filament directly below it, and it is customary to assume that the distribution is uniform.

Figure 122,II is part of an I beam section. At the plane where the web joins the flange, there must be a great difference in the intensity of the shearing stress. At *KL*, at some little distance down the web, the shearing stress becomes practically uniform over the section.

Formula XV may be used for beams having faces which are not parallel. The results are approximately correct.

Example

The section of a beam (Fig. 123,I) is an isosceles triangle of base 9 in. and altitude 12 in. Find the unit shearing stress at the neutral surface under a total shear of 6,480 lb.

$$s_s = \frac{6{,}480}{432 \times 6} \times \frac{8}{3} \times 24 = 160 \text{ psi}$$

Problems

71-1. In Fig. 123 find the unit shearing stress at the plane EF, which is midway between the vertex and the base. *Ans.* $s_s = 180$ psi.

The unit shearing stress is greater at the middle EF than at the neutral axis CD. The total shearing stress across EF is less than that at CD, but the ratio of the breadth at EF to the breadth at CD is still smaller.

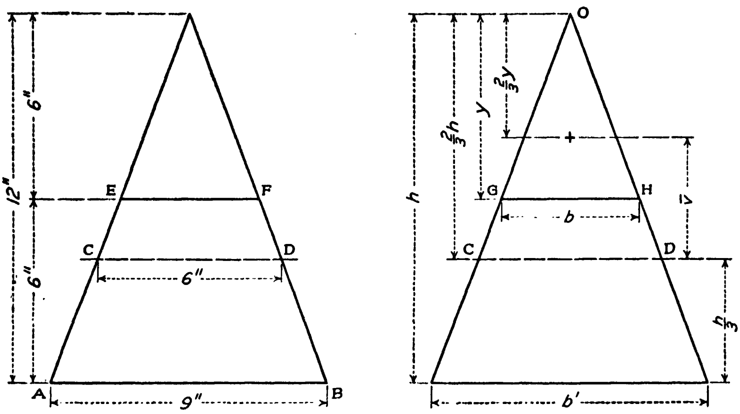


FIG. 123. Shearing stress in a triangular section.

Figure 123,II may be used to calculate the unit shearing stress at any plane GH at a distance y from the vertex of the beam. If b' is the base of the triangle, and b is the length of GH,

$$b = \frac{b'y}{h}; \qquad A' = \frac{b'y^2}{2h}; \qquad \bar{v} = \frac{2h}{3} - \frac{2y}{3} = \frac{2(h-y)}{3}; \qquad s_s = \frac{V}{3I}(hy - y^2)$$

The position of maximum unit shearing stress is given by

$$\frac{d}{dy}(hy - y^2) = 0 \qquad h - 2y = 0 \qquad y = \frac{h}{2}$$

The maximum unit shearing stress in a beam of triangular section with base horizontal is at the middle of the height.

71-2. Find the unit shearing stress at the neutral surface of a triangular beam and find the ratio of this stress to average unit stress. *Ans.* Ratio $= 4/3$.

71-3. For a beam of circular section, what is the ratio of the unit shearing stress at the neutral surface to the average vertical shearing stress?

Ans. Ratio $= 4/3$.

72. Shearing Stress in I Beams. It is customary to calculate the unit shearing stress in the web of an I beam by dividing the total vertical shear by the cross section of the web, which is regarded as extending the entire depth of the beam. If t is the thickness of the web and d is

the depth of the beam, this empirical formula is

$$\text{Unit shearing stress} = \frac{\text{total vertical shear}}{td} \qquad \text{Formula XVI}$$

In a 12-inch 31.8-pound I beam (Fig. 124), the thickness of the web is 0.35 inch; the area td is 4.2 square inches, and the average unit shearing stress, as computed by this method, is $0.238V$. To find the

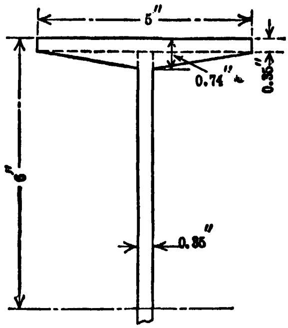


FIG. 124. I-beam section.

unit shearing stress at the neutral surface the upper half of the section is divided into a vertical rectangle, a horizontal rectangle, and two triangles, and the moment of each area is computed.

	A'	$\bar{v}$	Q
Horizontal rectangle	1.750	5.825	10.194
Two triangles	0.907	5.520	5.006
Vertical rectangle	1.977	2.825	5.558
Total	.....	.....	20.785

$$\frac{V}{Ib}Q = \frac{20.785V}{215.8 \times 0.35} = 0.275V$$

To find the unit shearing stress in a plane 5 inches above the neutral surface, which is a little below the flange, the moment of the vertical rectangle, 5 inches high and 0.35 inch thick, is subtracted from the moment of the entire upper half of the section.

$$Q = 20.785 - 5 \times 0.35 \times 2.5 = 20.785 - 4.375 = 16.410$$

$$s_s = \frac{16.410V}{215.8 \times 0.35} = 0.217V$$

The average of $0.217V$ under the top flange, $0.275V$ at the middle, and $0.217V$ above the bottom flange averages $0.236V$, which is very

close to the average shearing stress as computed by the empirical Formula XVI. However, it should be pointed out that Formula XV is used to compute what is defined in specifications as the maximum shearing stress in the webs of I beams.

Problems

72-1. Using the information above, compute the shearing stress by Formula XV at 1-in. intervals from the neutral axis for the 12-in. 31.8-lb I beam. Plot these values as abscissas and draw the shearing stress distribution curve.

72-2. A 14- by $6\frac{3}{4}$-in. wide-flange 34-lb I beam is supported at the ends and carries a concentrated load P at the middle of the span. Find the shearing stress at the neutral axis and at a point just under the flange. Sketch the shearing stress distribution curve. Solve also by Formula XVI.

Ans. $S_{s\max} = 0.276V$; $s_s = 0.213V$; $V/td = 0.248V$.

72-3. Find the unit shearing stress at the neutral axis and at 1.5 in. from the top for a 24- by 12-in. 100-lb wide-flange section, neglecting the fillets. Compare with V/td.

Ans. $s_s = 0.0984V$ and $0.0800V$; mean $= 0.0892V$; $V/(24 \times 0.468) = 0.0890V$.

72-4. A 10- by 8-in. wide-flange 33-lb I beam is supported at the ends and carries a concentrated load at the middle of the span, such that the maximum bending stress is 20,000 psi and the maximum shearing stress (by Formula XVI) is 12,000 psi. Find the load and the span. *Ans.* $L = 41$ in.

73. Failure of Beams. The nature of the failure in a beam depends principally upon the relative ultimate strength of the material in the

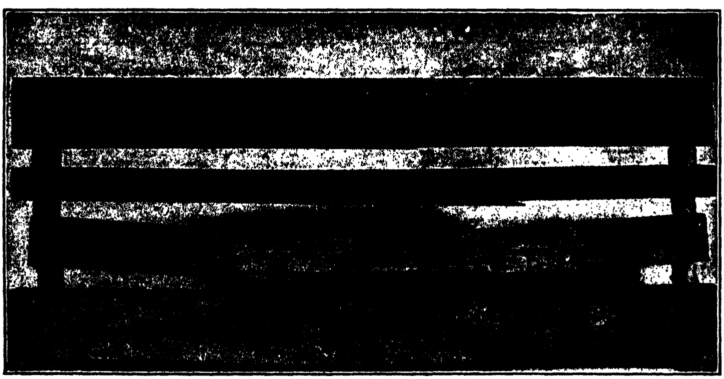


FIG. 125. Failure of timber beams.

different directions and the value of the different maximum stresses. In a beam which is short relative to its depth, the unit tensile and compressive stresses at the dangerous section are small compared with the unit shearing stress at the neutral surface at the ends. Owing to the fact that timber has a small shearing strength parallel to the grain,

such a beam, if made of timber, will usually fail by shear. Figure 125 shows four wooden beams each about 40 inches long. The upper beam is a yellow-pine beam glued to a white-pine beam. The total depth was 3.80 inches and breadth 1.57 inches. The beam was supported at points 36 inches apart and loaded at the third points; this beam failed by longitudinal shear at one end when the total load was 1,950 pounds. The failure followed the glued surface but began in the white pine.

Beams of brittle material, such as cast iron, hard steel, stone, or plain concrete, fail by tension. Soft-steel beams fail by buckling the compression flange or by buckling the web.

74. Shear Center. The restrictions underlying the theory of the flexure formula (Art. 66) included the assumption that the forces will

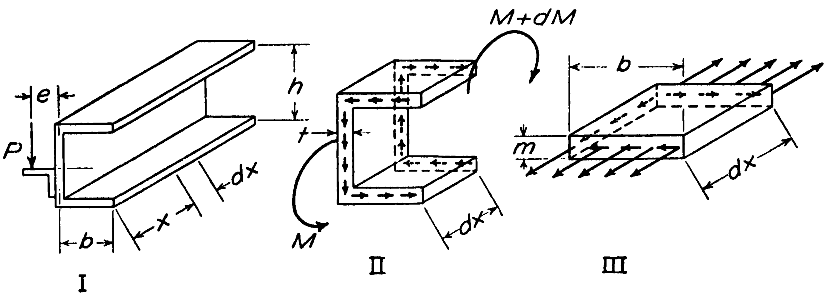


FIG. 126.

not cause twisting of the beam. Stresses were found in beams which had two axes of symmetry. In a few cases the beam had only one axis of symmetry, but in these cases the axis of symmetry was always in the plane of the loads. The channel laid flatwise so that the loads are perpendicular to the web is an example. Consider now the channel used as a cantilever as in Fig. 126,I with the load applied at the free end on a bracket attached to the web.

Figure 126,II shows the free body for the short elemental length dx. Neglecting the body weight, the body is subjected to a moment M at one end, and $M + dM$ at the other end, and also to shearing forces on the flanges and web at both ends. The top flange is shown as a free body in Fig. 126,III. There can be no shearing forces on the right (free) surface, but there is shear on the left face (where the flange joins the web) and on the front and rear surfaces. The bending stress develops a net tensile stress

$$S_t = \frac{dM\,c}{I} = \frac{dM\,h}{2I}. \tag{74.1}$$

Multiplying the tensile stress by the area and equating to the shearing force gives a resolution parallel to the length of the flange,

$$\frac{dM\,hmb}{2I} = S_s m\,dx \tag{74.2}$$

where S_s is considered uniform over the thickness of the flange. Remembering that $dM/dx = V$ and solving,

$$S_s = \frac{Vhb}{2I} \tag{74.3}$$

The shearing stress at junction of web and flange is given by Eq. (74.3). But the shearing stress on the rear face at the toe of the flange must be zero since there is no shear on the free edge. It is customary to assume that the stress varies directly with the distance along b. That this is a reasonable assumption can be verified by slicing the width b into a number of thin rods and observing that the bending (tensile) stress is assumed to be uniform over the end of the flange. The total shearing force on the top flange is the average stress times the area,

$$F = \frac{Vhmb^2}{4I} \tag{74.4}$$

There is also a shearing stress on the web but if moments are taken about it, this is eliminated. The moment of the couple formed by the flange shears will cause twisting of the channel unless the moment of the load, P, is equal and opposite.

$$Pe = \frac{Vh^2mb^2}{4I} \tag{74.5}$$

Using the length x as a free body, $V = P$ and

$$I = \frac{2mbh^2}{4} + \frac{th^3}{12} \tag{74.6}$$

neglecting the moment of inertia of the flange about its own axis. Substituting and solving for the position of the load,

$$e = \frac{\frac{1}{2}b}{1 + \dfrac{th}{6mb}} \tag{74.7}$$

Equation (74.7) gives the distance to the shear center. The principle can be extended to other sections such as angles, tees, etc., to find the shear center.

Problems

74-1. Find the shear center for a 10-in. 15.3-lb standard channel.
Ans. 0.90 in. from center of web.

74-2. Find the shear center for a 10-in. 30-lb standard channel.

75. Miscellaneous Problems

75-1. A 5- by 4-in. rectangular beam 8 ft long is supported at the left end and at 3 ft from the right end. The beam carries 4,000 lb on the right end and 6,000 lb at 2 ft from the left end. Neglect the weight of the beam and find (*a*) the maximum unit shearing stress; (*b*) the unit shearing stress at 1 in. below the top surface and 2 ft from the right end of the beam; and (*c*) the unit bending stress at the point described in (*b*). Indicate on your sketch of the beam the point where the stress in (*a*) occurs.

75-2. An 8- by $5\frac{1}{4}$-in. wide-flange 17-lb beam is supported at the ends on a 5-ft span. It carries a uniform load of 4,000 lb per ft over the 4 ft adjacent to the right end and no load on the remainder. Neglect the weight of the beam and find (*a*) the approximate average shearing stress at the section where it is greatest; (*b*) the unit shearing stress at a point 2 ft from the left end and 1 in. below the top surface of the beam; (*c*) the bending stress at the point named in (*b*). *Ans.* $s = 6{,}920$ psi.

75-3. A 4- by 6-in. rectangular beam 6 ft long is supported at the ends. Specifications for this timber limit the maximum bending stress to 1,200 psi and the maximum shearing stress to 100 psi. Find the maximum load which can be concentrated at 2 ft from the right end of the span.

75-4. An 8- by $6\frac{1}{2}$-in. wide-flange 24-lb I beam is supported at the ends and carries a uniform load, such that the maximum shearing stress does not exceed 12,000 psi and the maximum bending stress does not exceed 16,000 psi. Find the total safe load and the minimum span for which bending governs.

75-5. A 6- by 8-in. cantilever beam carries a uniform load over the outer half of its length and no load on the half-length near the wall. The maximum allowable bending stress is 1,200 psi and the maximum allowable shearing stress is 100 psi. Find the maximum load the beam may carry and the maximum length of beam consistent with these specifications.
Ans. $W = 3{,}200$ lb; $L = 32$ in.

75-6. In Fig. 119,II, two 4- by 3-in. beams are doweled together to make a beam 6 in. deep and 12 ft long. The beam carries a concentrated load at the middle of the span such that the maximum bending stress is 1,200 psi. The dowels are 1 in. in diameter and 3 in. long and made of timber which has an allowable shearing stress of 150 psi. How far apart should the dowels be spaced? *Ans.* 4.71 in.

CHAPTER 9

DEFLECTION OF BEAMS BY DOUBLE INTEGRATION

76. Relation of Stress to Deformation. Figure 127 represents a bent beam which is concave upward. *EFG* is a plane section with neutral axis *BB′*. The broken lines *K′M′*, *M′N′* indicate the position, before the beam was bent, of a plane section parallel to *EFG* at a distance Δl from that section. The section *EFG* may be regarded as

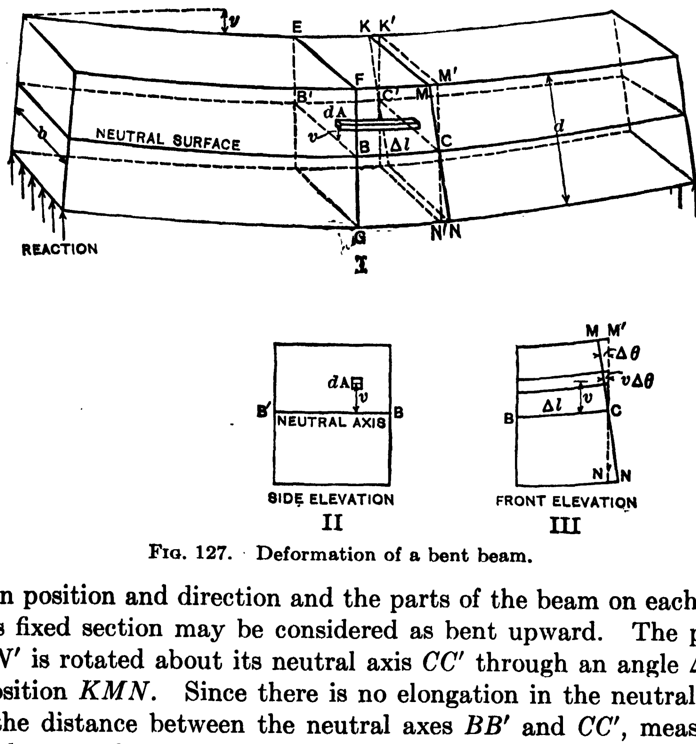


Fig. 127. Deformation of a bent beam.

fixed in position and direction and the parts of the beam on each side of this fixed section may be considered as bent upward. The plane *K′M′N′* is rotated about its neutral axis *CC′* through an angle $\Delta\theta$ to the position *KMN*. Since there is no elongation in the neutral surface, the distance between the neutral axes *BB′* and *CC′*, measured along the curved surface, remains unchanged.

It is assumed that a plane section in a beam remains plane when the beam is bent; therefore the plane section *K′M′N′* remains a plane section after it has been rotated into the position *KMN*. A filament of section *dA*, at a distance *v* from the neutral surface, extends from

the plane *EFG* to the plane *KMN*. When the beam is bent and the plane *KMN* is rotated about the neutral axis CC' through an angle $\Delta\theta$, this filament is shortened by an amount $v\,\Delta\theta$. A similar filament at a distance v below the neutral surface would be elongated $v\,\Delta\theta$. The unit deformation of the filament is given by

$$\epsilon = \frac{v\,\Delta\theta}{\Delta l} \tag{76.1}$$

Under the condition that the deformations are such that no stress exceeds the proportional elastic limit, the unit stress varies as the unit deformation; and since the unit deformation varies as v, the unit stress varies as the distance from the neutral axis, as was assumed in Art. 66.

The unit stress above the neutral surface is

$$s_c = E_c v \frac{\Delta\theta}{\Delta l} \tag{76.2}$$

Below the neutral surface,

$$s_t = E_t v \frac{\Delta\theta}{\Delta l} \tag{76.3}$$

In most cases *it is assumed* (and is practically true) *that the modulus of elasticity is the same in both compression and tension.* With this assumption,

$$s = Ev \frac{\Delta\theta}{\Delta l} \tag{76.4}$$

is the expression for the unit stress in the beam, at any element of area.

On an element of area,

$$\text{Total force on } dA = Ev \frac{\Delta\theta}{\Delta l}\, dA \tag{76.5}$$

The moment of this force with respect to the neutral axis BB' is the product of the total force on dA by the moment arm v.

$$dM = Ev^2 \frac{\Delta\theta}{\Delta l}\, dA \tag{76.6}$$

The total moment of all the filaments which make up the beam is the integral of dM over the section *EFG*. Integrating over this area, $\Delta\theta/\Delta l$ remains constant and

$$M = E\frac{\Delta\theta}{\Delta l}\int_{c_1}^{c_2} v^2\, dA = E\frac{\Delta\theta}{\Delta l}\, I \tag{76.7}$$

in which c_1, c_2 are the distances of the upper and lower surfaces of the

beam from the neutral surface, I is the moment of inertia of the cross section EFG or KMN with respect to its neutral axis, and $\Delta\theta$ is the change in slope, in the length Δl, of the normal to the beam or the equal change in slope of the tangent to the beam.

Example

A 6- by 6-in. wooden beam rests on two supports, which are 50 in. apart, and overhangs each support 30 in. Equal loads are placed on each overhanging end at 20 in. outside the supports. It is found that two sections between the supports, which are 40 in. apart and which were parallel to each other before the beam was loaded, make an angle of 1° with each other after the loads are applied. Find the bending moment between the supports and find the loads, if E is 1,500,000 psi.

By means of Eq. 76.7, find the bending moment between the supports.

$$I = 108 \text{ in.}^4 \qquad \Delta l = 40 \text{ in.} \qquad \Delta\theta = \frac{\pi}{180} \text{ rad.}$$

$$M = \frac{1{,}500{,}000 \times 108 \times \pi}{40 \times 180} = 22{,}500\pi = 70{,}686 \text{ in.-lb}$$

The bending moment is constant between the supports and equal to

$$20P = 70{,}686 \text{ in.-lb.}$$

Therefore, $P = 3{,}534$ lb.

Problems

76-1. A 5-in. 10-lb I beam rests on supports 6 ft apart and carries a load midway between the supports. Measurements are taken at the top and bottom of the beam on two 8-in. gage lengths. Gage length B begins 4 in. to the right of the middle, and gage length A starts from the right end of B. Neglecting the weight of the beam, what is the stress in the outer fibers at the middle of the beam when a load of 3,000 lb is applied? What is the average stress in the outer fibers for gage length B? For gage length A?
Ans. 11,250 psi: 8,750 psi; and 6,250 psi.

76-2. What is the maximum stress in the beam of Prob. 76-1 when the total load is 6,200 lb? *Ans.* 23,250 psi.

76-3. What is the change of average compressive stress in the upper fibers of gage length B when the load changes from 200 to 6,200 lb? In the lower fibers of B? In the upper fibers of A? In the lower fibers of A?
Ans. 17,500 and 12,500 psi.

76-4. In the beam of Prob. 76-1, when the load changes from 200 to 6,200 lb, the gage length B lengthens 0.00476 in. at the bottom and shortens the same amount at the top. Calculate E. *Ans.* $E = 29{,}410{,}000$ psi.

76-5. In the beam of Prob. 76-1, the gage length A shortens 0.00342 in. at the top and lengthens 0.00340 in. at the bottom when the load changes from 200 to 6,200 lb. Taking the average of these readings, calculate E.
Ans. $E = 29{,}300{,}000$ psi.

76-6. Calculate θ from the readings of Prob. 76-4 and solve for E by Eq. (76.7).

76-7. Calculate θ from the average of the readings of Prob. 76-5 and solve for E.

76-8. Through what angle may a steel plate be bent in a length of 20 in. if the

plate is 0.1 in. thick, E is 30,000,000 psi, and the elastic limit is 90,000 psi?
Ans. $90{,}000 = 30{,}000{,}000 \times 0.05\Delta\theta/\Delta l$; $\Delta\theta/\Delta l = 0.06$;
$\Delta\theta = 1.2$ rad $= 68°45'$.

76-9. A steel hacksaw blade 0.023 in. thick was bent 45° in a length of 4 in. by a constant moment. Find the unit stress in the outer fibers.

76-10. The elongation of an 8-in. gage length at the bottom of a 3-in. T section is 0.00150 in. and the compression of an equal gage length directly above at the top of the stem is 0.00453 in. How far is the neutral surface from the bottom of the flange? *Ans.* $y = \dfrac{3 \times 150}{603} = 0.746$ in.

76-11. A reinforced-concrete beam, 12 in. high, has one 20-in. gage length 1 in. from the top and another similar gage length 1 in. from the bottom. The upper gage shortens 0.0096 in., while the lower gage lengthens 0.0134 in. How far is the neutral surface from the extreme compression fibers? *Ans.* 5.18 in.

77. Relation of Moment to Curvature. In Eq. (76.7) it was shown that

$$M = EI\frac{\Delta\theta}{\Delta l} = EI\frac{d\theta}{dl} \tag{77.1}$$

for infinitesimal lengths measured along the neutral surface of the bent beam. The angle $d\theta$ is the change in slope of the tangent to the neutral surface in the length dl. In Figs. 127 and 128, two lines FG and MN

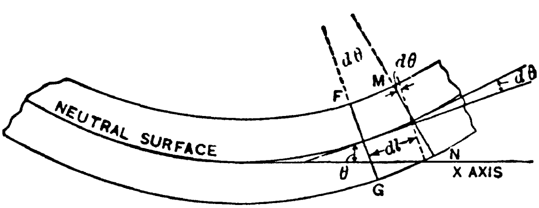


FIG. 128. Curvature of beam.

are drawn perpendicular to the neutral surface at a distance dl apart. The broken line which intersects MN at the neutral surface is parallel to FG. The lines FG and MN make an angle $d\theta$ with each other, since they are normal to the neutral surface and intersect at some point beyond the drawing, at a distance ρ from the neutral surface. This distance ρ is the radius of curvature of the neutral surface.

By geometry,

$$\rho\, d\theta = dl \tag{77.2}$$

$$\frac{d\theta}{dl} = \frac{1}{\rho} \tag{77.3}$$

Substituting in Eq. (77.1),

$$\frac{M}{EI} = \frac{1}{\rho} \qquad M = \frac{EI}{\rho} \tag{77.4}$$

If M is constant or if I varies as M, ρ is constant, and the curve of the beam is an arc of a circle which may be computed by trigonometry.

Problems

77-1. A 3- by 1-in. steel beam 10 ft long rests on two supports, each 30 in. from an end, and carries a load of 200 lb on each end (Fig. 129). If the weight of

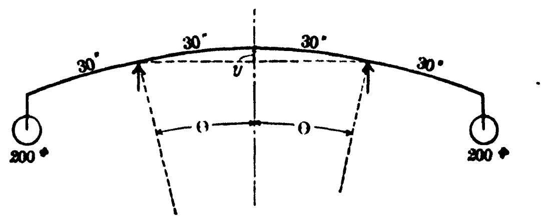


FIG. 129. Curvature constant.

the beam is neglected, what is the bending moment for the portion between the supports? If the modulus of elasticity is 30,000,000 psi, what is the radius of curvature? How much is the middle of the beam deflected upward above the supports? Solve for the deflection by geometry, assuming that the chord is equal to the arc. Solve also by trigonometry. If tables do not give the cosine with sufficient accuracy, calculate it by the series.

Ans. M = 6,000 in.-lb; ρ = 1,250 in.; y = 0.36 in.

77-2. A steel plate 1 in. wide and 0.1 in. thick is 30 in. long. Clamps at the ends bend the plate by constant moment until the maximum bending stress is 18,000 psi. Find the moment, the angle through which the plate is bent, the radius of curvature, and the deflection of the middle from the line joining the ends. *Ans.* 20°38′; y = 1.35 in.

77-3. A 4- by 6-in. timber beam 18 ft long rests on supports 10 ft apart and overhangs each support 4 ft. Calculate the radius of curvature for the portion between the supports when a load of 500 lb is placed on each end.

Ans. 300 ft.

77-4. A 12- by 1-in. aluminum beam 10 ft long is supported at 3 ft from each end. A 500-lb load is placed on each end. Neglecting the weight of the beam, find the radius of curvature of the portion between the supports. Find the maximum bending stress. *Ans.* S = 9,000 psi.

78. Change of Slope in Rectangular Coordinates. When Eq. (77.1) is integrated with θ and l as the variables, the integral gives the change of slope in radians. For most purposes, it is desirable to express the slope in rectangular coordinates. The curved line of Fig. 130 represents the neutral surface of a bent beam with the deflection greatly exaggerated. In a floor beam, for instance, the deflection

should not be more than 1 inch in 30 feet. A deflection as great as would be permissible in an engineering structure would not be noticeable in a drawing. For this reason, in the discussion of deflection which follows, a beam has been drawn in each figure as it would appear to the eye, and then a heavy line has been drawn below to represent the position and slope of the neutral surface with all vertical distances magnified.

In Fig. 130, dl is a small length measured along the neutral surface, and dx is the projection of this length on the horizontal. $dx = dl \cos \theta$,

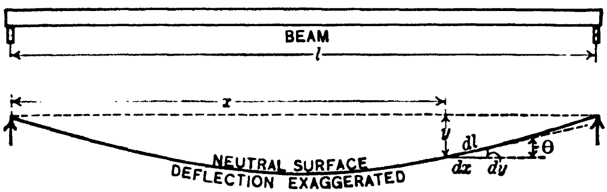


FIG. 130. Slope and deflection.

in which θ is the slope of the tangent to the beam. With the small deflections allowable in an engineering structure, $\cos \theta$ is nearly unity and dx is practically equal to dl. For instance, if the slope were 1 part in 100 (which is relatively large) a triangle could be drawn with a base of 1 unit and an altitude of 0.01 unit. The hypotenuse of this triangle is $\sqrt{1.0001}$, which is 1.00005, and the error in the assumption that dx equals dl is 1 part in 20,000. When dx is substituted for dl, Eq. (77.1) becomes

$$M = EI \frac{d\theta}{dx} \tag{78.1}$$

Since θ is a small angle, its value in radians is practically equal to its tangent.

$$\theta = \tan \theta = \frac{dy}{dx} \tag{78.2}$$

$$\frac{d\theta}{dx} = \frac{d^2y}{dx^2} \tag{78.3}$$

$$M = EI \frac{d^2y}{dx^2} \qquad \text{Formula XVII}$$

This formula may be derived in a slightly different manner, which will show the magnitude of the approximations. From calculus, the reciprocal of the radius of curvature is

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \tag{78.4}$$

When this expression is substituted in Eq. (77.4) the result is

$$M = \frac{EI\, d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \tag{78.5}$$

When dy/dx is small, $(dy/dx)^2$ is much smaller and may be neglected. The denominator of the second member of Eq. (78.5) then becomes unity, and Eq. (78.5) is equivalent to Formula XVII.

In some problems in which the deflections are large, the exact formula of Eq. (78.5) is required. For all ordinary problems of beam and column deflection, Formula XVII is ample. In the application of the formula, the X axis is taken parallel to the unbent beam, the Y axis is taken positive upward, and the deflection is so small that $(dy/dx)^2$ is negligible compared with unity.

79. Solution of the Differential Equation of Deflection. Before solving Formula XVII for the deflection of a beam or column, all the factors must be expressed in terms of x, y, and *constants*. The modulus of elasticity is constant, provided the unit stress does not exceed the proportional elastic limit. The formulas for deflection are valid only below this limit. For beams of uniform section, I is constant; for beams of variable section, it is expressed as a function of x. The moment is expressed as a function of x and y. In beams it is usually a function of x only.

When the expressions for M and I do not depend upon the deflection y, Formula XVII becomes

$$\frac{d^2y}{dx^2} = \text{function of } x \tag{79.1}$$

The first integration of Eq. (79.1) gives dy/dx as a function of x with the addition of an integration constant. If dy/dx is known for any one value of x, these values may be substituted in the integral and the integration constant determined.

The second integration gives the deflection y as a function of x with the addition of a second integration constant. If y is known for some one value of x, these values may be substituted in the second integral and the second integration constant determined. The final integral with the integration constants given in terms of the loads and dimensions of the beam is called the *equation of the elastic line*.

If dy/dx is not known for any one value of x, the first integration constant must be carried through the second integration, and the value of y must be known for two values of x to complete the solution. The next eight articles will illustrate the method.

80. Cantilever Loaded at Free End. Figure 131 represents a cantilever beam fixed at the right end and loaded at the left end. The beam was horizontal before the load was applied and remains horizontal at the wall when loaded. The origin of coordinates is taken at the posi-

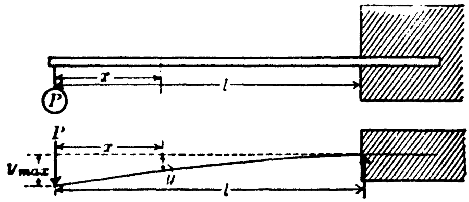


FIG. 131. Cantilever with load on free end.

tion of the left end before the load was applied. The moment at *any* distance x from the origin is $-Px$. The differential equation is

$$EI\frac{d^2y}{dx^2} = -Px \tag{80.1}$$

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + C_1 \tag{80.2}$$

At the wall, where $x = l$ the beam is horizontal and $dy/dx = 0$.

$$C_1 = \frac{Pl^2}{2} \tag{80.3}$$

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + \frac{Pl^2}{2} \tag{80.4}$$

$$EIy = -\frac{Px^3}{6} + \frac{Pl^2x}{2} + C_2 \tag{80.5}$$

At the wall $x = l$, $y = 0$.

$$0 = -\frac{Pl^3}{6} + \frac{Pl^3}{2} + C_2$$

$$C_2 = -\frac{Pl^3}{3} \tag{80.6}$$

$$EIy = -\frac{Px^3}{6} + \frac{Pl^2x}{2} - \frac{Pl^3}{3} \tag{80.7}$$

$$y = -\frac{P}{6EI}(2l^3 - 3l^2x + x^3) \tag{80.8}$$

The maximum deflection is at the free end, where $x = 0$.

$$y_{max} = -\frac{Pl^3}{3EI} \tag{80.9}$$

If $x = kl$, in which k is a fraction less than unity,

$$y = -\frac{Pl^3}{6EI}(2 - 3k + k^3) \tag{80.10}$$

Problems

80-1. A 4- by 6-in. cantilever 10 ft long carries a load of 240 lb on the free end. Find the deflection at the free end if $E = 1{,}500{,}000$ psi. Find the maximum unit stress. *Ans.* $y_{max} = -1.28$ in.; $S = 1{,}200$ psi.

80-2. What would be the deflection at the end and the maximum stress for the beam of Prob. 80-1 if it were turned 90° to bring the 4-in. faces vertical?

80-3. A 10-in. 25.4-lb standard steel I beam deflects 0.34 in. at the free end when a load of 3,660 lb is placed on the end of an 8-ft 4-in. length. Find E and the maximum bending stress. *Ans.* $E = 29{,}400{,}000$ psi.

80-4. Find the deflection of the beam of Prob. 80-3 at 30 in. from the free end. *Ans.* -0.192 in.

80-5. For the beam of Prob. 80-3, find the slope at the free end and at a point 30 in. from the free end. *Ans.* $dy/dx = 0.00464$.

80-6. An I beam is used as a cantilever. Derive an equation for the maximum deflection in terms of the maximum bending stress, length, depth, and modulus of elasticity. *Ans.* $y_{max} = -2Sl^2/3Ed$.

80-7. Find the deflection at the end of any 10-in. steel I beam which is stressed to the allowable bending stress, if the length is 7 ft 6 in. *Ans.* -0.36 in.

80-8. A timber beam in the form of an isosceles triangle 6 in. deep has the base of the triangle parallel to the neutral axis of the beam. If the length is 10 ft and the beam carries a load on the free end which stresses it to the allowable bending stress, find the maximum deflection. *Ans.* -1.2 in.

80-9. An aluminum T is 4 by 5 by ½ in. weighing 5.56 lb per ft. Neglect the weight of the beam and find the maximum deflection when a 600-lb load is placed on the free end of a 5-ft cantilever. Find the maximum bending stress. *Ans.* $y = -0.375$ in.; $S = 11{,}500$ psi.

80-10. A cantilever of length l is fixed at the left end and carries a load P at the right end. With the origin of coordinates at the fixed end and x positive toward the right, derive the equation of the elastic line. Draw a sketch of the elastic line showing the coordinates to a section, as in Fig. 131.

$$\textit{Ans. } M = -P(l - x) = -Pl + Px \tag{80.11}$$

$$EI\frac{dy}{dx} = -Plx + \frac{Px^2}{2} + [C_1 = 0] \tag{80.12}$$

$$EIy = -\frac{Plx^2}{2} + \frac{Px^3}{6} + [C_2 = 0] \tag{80.13}$$

80-11. Solve Prob. 80-10 using the first expression for M in Eq. (80.11).

$$\textit{Ans. } EI\frac{dy}{dx} = \frac{P(l - x)^2}{2} - \frac{Pl^2}{2} \tag{80.14}$$

$$EIy = -\frac{P(l - x)^3}{6} - \frac{Pl^2x}{2} + \frac{Pl^3}{6} \tag{80.15}$$

81. Cantilever with Load at Any Point. Figure 132 shows a cantilever, which is fixed at the right end and carries a concentrated load P at a distance a from the left end. The portion to the left of the load remains straight (if the weight of the beam is neglected). The portion

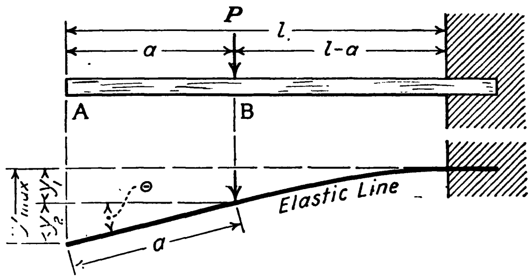


FIG. 132.

to the right of the load is a cantilever of length $l - a$ which is loaded at the end. The deflection of this portion is

$$y_1 = -\frac{P(l-a)^3}{3EI} \tag{81.1}$$

The additional deflection of the portion to the left of the load is $y_2 = -a \sin \theta$, in which θ is the slope at the load. For a small angle, $\sin \theta = \tan \theta = dy/dx$. From Eq. (80.4), the slope under the load is

$$\frac{dy}{dx} = \frac{P(l-a)^2}{2EI} \tag{81.2}$$

$$y_2 = -\frac{Pa(l-a)^2}{2EI}$$

$$y_{max} = -\frac{P(l-a)^3}{3EI} - \frac{Pa(l-a)^2}{2EI} = -\frac{P}{6EI}(2l^3 - 3l^2a + a^3) \tag{81.3}$$

Problems

81-1. A 10-in. 25.4-lb standard steel I beam 8 ft 4 in. long carries a 3,660-lb load at 30 in. from the free end. If $E = 29{,}400{,}000$ psi, find the deflection at the free end. Check with Prob. 80-4.

81-2. The beam of Prob. 81-1 also carries a 3,660-lb load on the free end. Using previous answers, find the maximum deflection.

Ans. -0.532 in. (What about the stress?)

81-3. A 6- by 4-in. timber beam 6 ft long carries 200 lb at 1 ft from the free end. Find the deflection at the free end. The 200-lb load is removed and a 120-lb load applied at 2 ft from the free end. Find the deflection at the free end. The 200-lb load is now returned to its position. Find the deflection at the free end caused by both loads.

82. Maxwell's Theorem. As long as the stress remains below the proportional elastic limit, the deflection caused by several loads is equal to the sum of the deflections which would be caused by the loads acting separately. This statement, which is called the principle of *superposition,* may be regarded as an axiom amply verified by experiment and by theoretical investigation of special cases. The theorem is derived later by energy methods. This law of the equivalence of reciprocal deflections is: *If A and B are two points on a beam (or any elastic structure), the deflection at A caused by a given load at B is equal to the deflection at B caused by the same load at A.*

83. Cantilever with Load Uniformity Distributed. Figure 133 shows a cantilever which is fixed at the right end and carries a load of

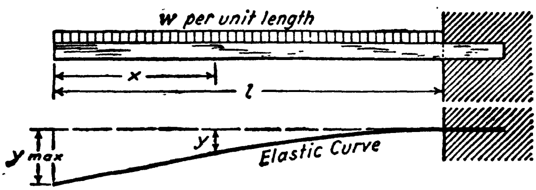


FIG. 133. Cantilever with distributed load, fixed at right end.

w per unit length. The origin of coordinates is taken at the position of the free end before the load was applied.

The moment at any section at a distance x from the free end is $-wx^2/2$.

$$EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2} \tag{83.1}$$

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1 \tag{83.2}$$

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wl^3}{6} \tag{83.3}$$

$$EIy = -\frac{wx^4}{24} + \frac{wl^3x}{6} + C_2 \tag{83.4}$$

$$EIy = -\frac{wx^4}{24} + \frac{wl^3x}{6} - \frac{wl^4}{8} \tag{83.5}$$

$$y = -\frac{w}{24EI}(3l^4 - 4l^3x + x^4) \tag{83.6}$$

which is the equation of the elastic line.

If $x = kl$, in which k is a fraction smaller than unity, Eq. (83.6) may be written

$$y = -\frac{wl^4}{24EI}(3 - 4k + k^4) = -\frac{Wl^3}{24EI}(3 - 4k + k^4) \tag{83.7}$$

At the free end

$$y_{max} = -\frac{wl^4}{8EI} = -\frac{Wl^3}{8EI} \tag{83.8}$$

In these equations $W = wl$, which is the total distributed load.

Since modulus of elasticity is expressed in pounds per square inch, and moment of inertia in inches^4, w must be in pounds per inch. To find the total load W, pounds per foot may be used with length in feet.

Problems

83-1. A 6- by 10-in. timber cantilever, 10 ft long, is subjected to a load of 180 lb per ft. What is the deflection at the free end if $E = 1{,}600{,}000$ psi? What is the maximum fiber stress at the fixed end? Use the total load W.

Ans. $y_{max} = -0.486$ in.; $S = 1{,}080$ psi.

83-2. In Prob. 83-1, what is the deflection 3 ft from the free end?

Ans. $y = -0.2929$ in.

83-3. A 2- by 3-in. timber beam weighing 24 lb has 2 ft of its length clamped between horizontal steel plates and projects 10 ft as a cantilever. The deflection caused by its weight brings the lower edge at the free end 0.48 in. below the plane of the upper surface of the lower steel plate. Assuming that the beam was originally straight, find E. *Ans.* $E = 2{,}000{,}000$ psi.

83-4. Assuming that the beam of Prob. 83-3 is turned 180° and the deflection of the free end is apparently 0.52 in., calculate the corrected E.

83-5. If a load of 25 lb on the beam of Prob. 83-3 at 1 ft 8 in. from the free end produces an additional deflection of 0.81 in., find E.

83-6. A cantilever with uniformly distributed load is fixed at the left end, as in Fig. 134. Show that

$$M = -\frac{w(l-x)^2}{2} \tag{83.9}$$

$$EI\frac{dy}{dx} = \frac{w(l-x)^3}{6} - \frac{wl^3}{6} \tag{83.10}$$

$$EIy = -\frac{w(l-x)^4}{24} - \frac{wl^3x}{6} + \frac{wl^4}{24} \tag{83.11}$$

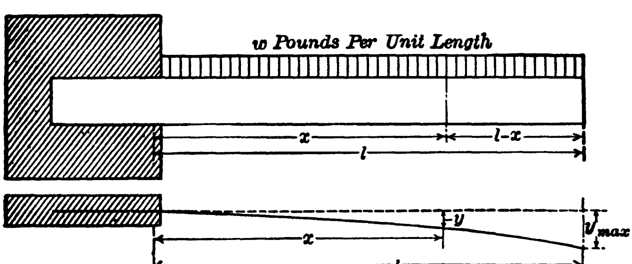


FIG. 134. Cantilever with uniformly distributed load.

83-7. Expand the expression for M in Eq. (83.9) and integrate for the equation of the elastic line.

$$\textit{Ans.}\ EIy = -\frac{wl^2x^2}{4} + \frac{wlx^3}{6} - \frac{wx^4}{24} \tag{83.12}$$

83-8. A cantilever beam is fixed at the right end with a slight upward inclination toward the left (like a springboard) such that, when a uniform load is applied, the beam deflects and becomes horizontal at the free end. Derive the equation of the elastic line, taking an origin with the Y axis through the free end, and the X axis horizontally through the wall where the beam joins the wall.

Ans. $y = \frac{w}{24EI}(l^4 - x^4)$.

83-9. The timber beam of Prob. 83-8 is a 12- by 2-in. springboard 10 ft long. When a total uniform load of 200 lb is applied, the free end of the beam takes a horizontal slope. What is the elevation of the free end above the horizontal line through the fixed end of the beam? What was the original slope of the unloaded beam? How much did the free end of the beam deflect under the load? What is the maximum bending stress in the beam?

Ans. 1.5 in.; 0.05 = 2°52′; 4.5 in.; 1,500 psi.

NOTE: With this small slope, very little error is introduced in using the length of the beam equal to its horizontal projection.

83-10. A cantilever beam fixed at the right end has a slight upward slope to the left (like a springboard). When a uniform load is applied the beam deflects so that the free end lies on the same horizontal line as the fixed end. Derive the equation of the elastic line, using the free end as the origin.

Ans. $y = wx(l^3 - x^3)/24EI$.

83-11. The timber beam of Prob. 83-10 is a 9- by 4-in. springboard 5 ft long. When a uniform load of 240 lb per ft is applied, the free end deflects to the same horizontal line as the fixed end. Find the original slope of the unloaded beam. Find where the beam is horizontal. Find the maximum bending stress in the beam. *Ans.* 0.009375 rad; $x = 0.63l$; $S = 1,500$ psi.

84. Simply Supported Beam, Uniformly Loaded. In a beam supported at the ends and uniformly loaded, each reaction is one-half the

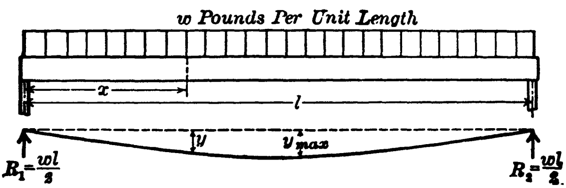


FIG. 135. Supports at ends, load uniformly distributed.

total load wl. The moment at a distance x from the left support is

$$M = \frac{wlx}{2} - \frac{wx^2}{2} \tag{84.1}$$

and the differential equation becomes

$$EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2} \tag{84.2}$$

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + C_1 \tag{84.3}$$

From symmetry, it is evident that the maximum deflection is at the middle.

$$\frac{dy}{dx} = 0 \qquad \text{when } x = \frac{l}{2}$$

$$C_1 = -\frac{wl^3}{24} \tag{84.4}$$

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24} \tag{84.5}$$

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24} + C_2 \tag{84.6}$$

At $x = 0$, $y = 0$, $C_2 = 0$,

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24} \tag{84.7}$$

When $x = l/2$ the deflection is a maximum,

$$y_{max} = -\frac{5wl^4}{384EI} = -\frac{5Wl^3}{384EI} \tag{84.8}$$

Substituting $x = l$ in Eq. (84.7), the deflection at the right support is found to be zero. This condition might have been used to determine one of the constants.

Problems

84-1. A 2- by 12-in. timber floor joist is 15 ft long between supports and carries a distributed load of 180 lb per ft. What is the deflection at the middle and at 5 ft from each end? What is the maximum fiber stress?
Ans. $y_{max} = -0.593$ in.

84-2. The floor of a room 30 ft long is carried on 24-in. I beams spaced 12 ft apart. The total dead load and live load is 250 lbs. per square foot. Select the I beam. Calculate the maximum deflection for this I beam. *Ans.* -0.648 in.

84-3. An 8-in. 18.4-lb standard steel I beam 12 ft long is supported at the ends and carries a load of 1,200 lb per ft including its own weight. Find the deflection at the middle and the slope at the left end. Find the maximum bending stress. *Ans.* $y_{max} = -0.328$; $dy/dx = -0.00728$.

84-4. An 8-in. 6.53-lb aluminum I beam 12 ft long is supported at the ends and carries a uniform load (including the weight of the beam) so that the maximum deflection is the same as for the steel I beam in Prob. 84-3. Find the uniform load. Find the maximum bending stress. *Ans.* 429 lb per ft.

84-5. A beam of length l and depth d has its neutral surface midway between the top and bottom. The beam is supported at the ends and carries a uniformly distributed load which makes the maximum stress equal to S. What is the deflection at the middle? *Ans.* $y_{max} = -5Sl^2/24Ed$.

84-6. Find the maximum deflection of a 12-in. steel I beam 15 ft long which is stressed to the allowable bending stress. *Ans.* -0.375 in.

84-7. A 1- by ¼-in. steel bar 10 ft long, supported at the ends, is deflected 5 in. by its weight. Find E. Find maximum stress. *Ans.* $S = 12{,}240$ psi.

84-8. A beam is supported at the ends, but the beam is not horizontal. The right support is slightly higher than the left, so that when the uniform load is applied the beam sags and just becomes horizontal over the left support. Taking an origin at the left support, derive the equation of the elastic line.

Ans. $y = \dfrac{wlx^3}{12EI} - \dfrac{wx^4}{24EI}$.

84-9. A 6- by 4-in. rectangular wood beam 10 ft long is supported at the ends with the right support slightly higher than the left, so that when loaded with a uniformly distributed load of 128 lb per ft the left end of the beam is horizontal over the support. Find how high the right support must be above the left to make this possible. Find the maximum bending stress.

Ans. 2.40 in.

85. Simply Supported Beam, Loaded at Middle. If P is the load at the middle, each reaction is $P/2$, and the moment from the left end to the middle is $Px/2$. For the portion of the beam between the left end and the middle,

$$EI\frac{d^2y}{dx^2} = \frac{Px}{2} \tag{85.1}$$

$$EI\frac{dy}{dx} = \frac{Px^2}{4} + C_1 \tag{85.2}$$

At the middle, from the symmetry of the sides, $dy/dx = 0$.

$$C_1 = -\frac{Pl^2}{16} \tag{85.3}$$

$$EI\frac{dy}{dx} = \frac{Px^2}{4} - \frac{Pl^2}{16} \tag{85.4}$$

$$EIy = \frac{Px^3}{12} - \frac{Pl^2x}{16} + C_2 \tag{85.5}$$

At the left support, where $x = 0$, $y = 0$,

$$C_2 = 0 \tag{85.6}$$

$$EIy = \frac{Px^3}{12} - \frac{Pl^2x}{16} \tag{85.7}$$

At the middle, where $x = l/2$,

$$y_{\max} = \frac{Pl^3}{96EI} - \frac{Pl^3}{32EI} = -\frac{Pl^3}{48EI} \tag{85.8}$$

Since the moment equation applies only to the left half of the beam, the formulas derived from this equation are not valid beyond the middle. From the middle to the right end, the moment is $\dfrac{P}{2}(l - x)$.

$$EI\frac{d^2y}{dx^2} = \frac{P}{2}(l - x) \tag{85.9}$$

$$EI\frac{dy}{dx} = -\frac{P(l - x)^2}{4} + \left(C_3 = \frac{Pl^2}{16}\right) \tag{85.10}$$

$$EIy = \frac{P(l - x)^3}{12} + \frac{Pl^2x}{16} + C_4 \tag{85.11}$$

At the right support, where $x = l$, $y = 0$,

$$C_4 = -\frac{Pl^3}{16}$$

$$EIy = \frac{P(l - x)^3}{12} + \frac{Pl^2x}{16} - \frac{Pl^3}{16} \tag{85.12}$$

A beam supported at the ends and loaded at the middle may be regarded as two cantilevers, each of length $l/2$, which are bent upward by reactions $P/2$.

Equation (85.8) is largely used in the determination of the modulus of elasticity.

Problems

85-1. A 4- by 6-in. beam, 5 ft long between supports, is deflected 0.0600 in. at the middle when the load changes from 200 to 1,400 lb. Find E and the maximum stress. *Ans.* E = 1,250,000 psi; S = 875 psi.

85-2. A timber beam 1.5 by 2 in. resting on supports 20 in. apart and loaded at the middle deflected 0.0625 in. when the load increased from 100 to 700 lb. The beam failed under a center load of 1,960 lb. Find the modulus of elasticity and the modulus of rupture. *Ans.* 1,600,000 psi; 9,800 psi.

85-3. The cast-iron T beam described in Prob. 67-3 was tested on a 20-in. span. When the center load changed from 200 to 800 lb, the center deflection increased 0.040 in. Find E. *Ans.* 15,625,000 psi.

85-4. A 6-in. 4.43-lb aluminum I beam 8 ft long is supported at the left end and at 3 ft from the right end. A gage is placed under the center of the 5-ft span. How much will the gage read when a 600-lb load is applied in the middle of the span? How much will the right end of the beam elevate?
Ans. −0.0115; +0.0208 in.

85-5. If S is the allowable unit stress in a beam, d is the depth, and l is the length, and if the neutral surface is midway between the top and bottom, what is the expression for the maximum allowable deflection when the beam is loaded at the middle? By means of this formula find the maximum allowable deflection in a steel beam 20 ft long and 18 in. deep.

86. Beam with Constant Moment. Figure 136 shows a beam which is supported at two points at a distance l apart, overhangs each support, and carries a load on each end. The moment at the left support is $-Pa$, and the moment at the right support is $-Qb$. If $Pa = Qb$, the left reaction is equal to P and the right reaction is equal to Q. The

loads and reactions then form two equal and opposite couples, and the moment between the supports is constant and equal to $-Pa$(or $-Qb$). With the origin of coordinates at the left support,

$$EI\frac{d^2y}{dx^2} = -Pa \tag{86.1}$$

$$EI\frac{dy}{dx} = -Pax + C_1 \tag{86.2}$$

$$EIy = -\frac{Pax^2}{2} + C_1x + C_2 \tag{86.3}$$

At the left support, where $x = 0$, $y = 0$; hence $C_2 = 0$.

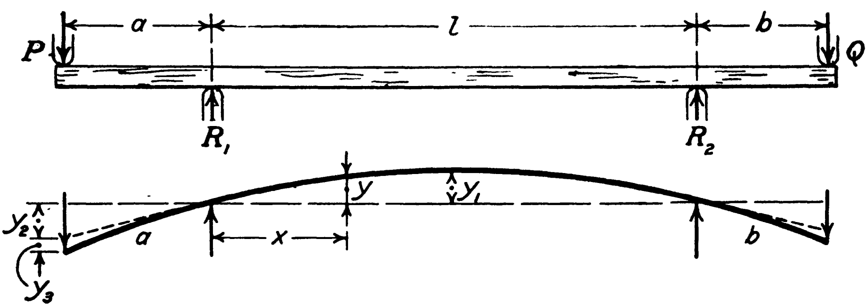


FIG. 136. Moment constant.

At the right support, where $x = l$, $y = 0$.

$$0 = -\frac{Pal^2}{2} + C_1l$$

$$C_1 = \frac{Pal}{2}$$

When this value of C_1 is substituted in Eq. (86.2),

$$EI\frac{dy}{dx} = -Pax + \frac{Pal}{2} \tag{86.4}$$

At the point of maximum deflection, $dy/dx = 0$ and $x = l/2$. The maximum deflection is at the middle of the span.

The equation of the elastic line is

$$y = \frac{Pa}{2EI}(lx - x^2) = \frac{Pax}{2EI}(l - x) \tag{86.5}$$

$$y_{max} = \frac{Pal^2}{8EI} = -\frac{M'l^2}{8EI} \tag{86.6}$$

in which M' is a positive, constant moment.

If P and Q are equal and a and b are equal, it is evident that the elastic line is symmetrical with respect to the middle of the span. Under these conditions, it could be assumed that dy/dx is zero when x is $l/2$, and the value of C_1 could be obtained from the first integral. If P and Q are not equal but the product Pa equals Qb, the symmetry is not self-evident, and it is better to determine both constants from the second integral.

The overhanging ends of Fig. 136 are cantilevers. The deflection of each load *from the tangent at the support* (y_3 of Fig. 136) is given by Eq. (80.9). The deflection of the tangent at the support from the horizontal line through the supports (y_2 of Fig. 136) is the distance from the load to the support multiplied by the slope at the support.

$$y_2 + y_3 = -\frac{Pa^2l}{2EI} - \frac{Pa^3}{3EI} = -\frac{Pa^2}{6EI}(3l + 2a) \qquad (86.7)$$

Problems

86-1. A board 6 in. wide and 1 in. thick rests on two supports 80 in. apart. A load of 30 lb is placed 16 in. to the left of the left support and a load of 20 lb is placed 24 in. to the right of the right support. If E is 1,200,000 psi, what is the deflection upward midway between the supports? What is the slope at each support? *Ans.* $y = 0.64$ in.; $dy/dx = 0.032$.

86-2. In Prob. 86-1, find the deflection at 20 in. to the right of the left support.

86-3. In Prob. 86-1, find the radius of curvature of the portion of the beam between the supports.

86-4. In Prob. 86-1, find the deflection of the load of 30 lb downward from the horizontal line through the supports. *Ans.* $-0.512 - 0.068 = -0.580$ in.

87. Simply Supported Beam, Loaded at Any Point. Since the moment equation changes at the concentrated load, two differential equations of the second order

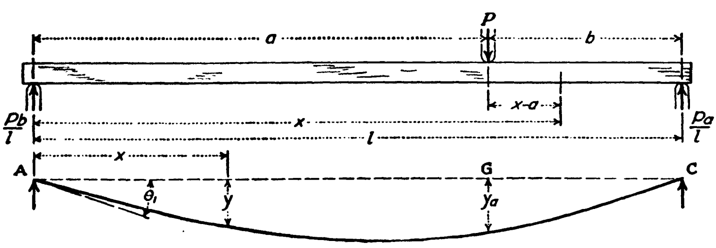


FIG. 137. Beam with single concentrated load.

must be solved to obtain the equation of the elastic line for the entire beam. For a beam supported at the ends with a load at the middle, the symmetry made it possible to obtain the integration constants from the single equation of the left half. Where the load is not at the middle, the slope is known neither at the middle nor under the load. If the moment equation for the portion of the beam between the left support of Fig. 137 and the load were integrated alone, the only known

loads and reactions then form two equal and opposite couples, and the moment between the supports is constant and equal to $-Pa$(or $-Qb$). With the origin of coordinates at the left support,

$$EI\frac{d^2y}{dx^2} = -Pa \tag{86.1}$$

$$EI\frac{dy}{dx} = -Pax + C_1 \tag{86.2}$$

$$EIy = -\frac{Pax^2}{2} + C_1x + C_2 \tag{86.3}$$

At the left support, where $x = 0$, $y = 0$; hence $C_2 = 0$.

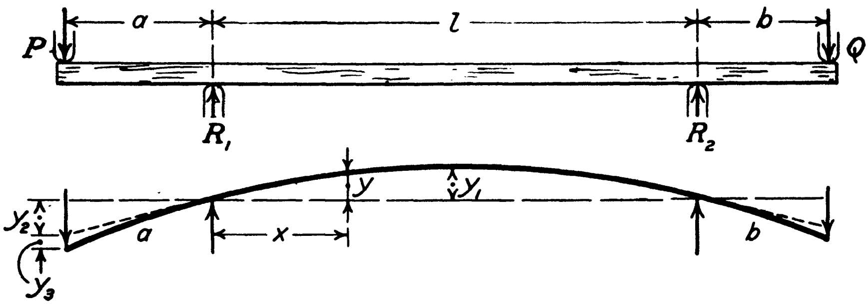


FIG. 136. Moment constant.

At the right support, where $x = l$, $y = 0$.

$$0 = -\frac{Pal^2}{2} + C_1l$$

$$C_1 = \frac{Pal}{2}$$

When this value of C_1 is substituted in Eq. (86.2),

$$EI\frac{dy}{dx} = -Pax + \frac{Pal}{2} \tag{86.4}$$

At the point of maximum deflection, $dy/dx = 0$ and $x = l/2$. The maximum deflection is at the middle of the span.

The equation of the elastic line is

$$y = \frac{Pa}{2EI}(lx - x^2) = \frac{Pax}{2EI}(l - x) \tag{86.5}$$

$$y_{\max} = \frac{Pal^2}{8EI} = -\frac{M'l^2}{8EI} \tag{86.6}$$

in which M' is a positive, constant moment.

If P and Q are equal and a and b are equal, it is evident that the elastic line is symmetrical with respect to the middle of the span. Under these conditions, it could be assumed that dy/dx is zero when x is $l/2$, and the value of C_1 could be obtained from the first integral. If P and Q are not equal but the product Pa equals Qb, the symmetry is not self-evident, and it is better to determine both constants from the second integral.

The overhanging ends of Fig. 136 are cantilevers. The deflection of each load *from the tangent at the support* (y_3 of Fig. 136) is given by Eq. (80.9). The deflection of the tangent at the support from the horizontal line through the supports (y_2 of Fig. 136) is the distance from the load to the support multiplied by the slope at the support.

$$y_2 + y_3 = -\frac{Pa^2l}{2EI} - \frac{Pa^3}{3EI} = -\frac{Pa^2}{6EI}(3l + 2a) \qquad (86.7)$$

Problems

86-1. A board 6 in. wide and 1 in. thick rests on two supports 80 in. apart. A load of 30 lb is placed 16 in. to the left of the left support and a load of 20 lb is placed 24 in. to the right of the right support. If E is 1,200,000 psi, what is the deflection upward midway between the supports? What is the slope at each support? *Ans.* $y = 0.64$ in.; $dy/dx = 0.032$.

86-2. In Prob. 86-1, find the deflection at 20 in. to the right of the left support.

86-3. In Prob. 86-1, find the radius of curvature of the portion of the beam between the supports.

86-4. In Prob. 86-1, find the deflection of the load of 30 lb downward from the horizontal line through the supports. *Ans.* $-0.512 - 0.068 = -0.580$ in.

87. Simply Supported Beam, Loaded at Any Point. Since the moment equation changes at the concentrated load, two differential equations of the second order

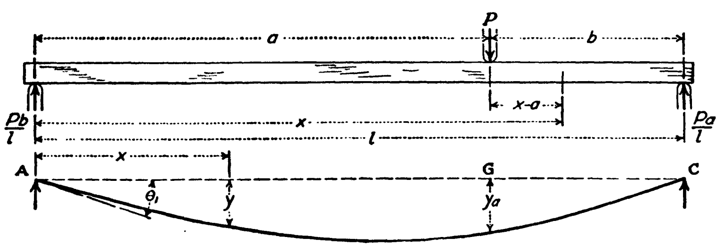


FIG. 137. Beam with single concentrated load.

must be solved to obtain the equation of the elastic line for the entire beam. For a beam supported at the ends with a load at the middle, the symmetry made it possible to obtain the integration constants from the single equation of the left half. Where the load is not at the middle, the slope is known neither at the middle nor under the load. If the moment equation for the portion of the beam between the left support of Fig. 137 and the load were integrated alone, the only known

condition would be that $y = 0$ when $x = 0$. If the equation for the portion between the load and the right support were integrated alone, the known condition would be $y = 0$ when $x = l$. Since the solution of these two equations introduces four integration constants, two other relations must be known. When the two moment equations are integrated together, these two additional relations may be found from the fact that under the load, where $x = a$, the slope dy/dx is the same for the two first integrals, and the deflection y is the same for the two second integrals. Since the principle of this method is important, it will be carried through, although briefer solutions are given later.

Figure 137 represents a beam of length l, which is supported at the ends and carries a load P at a distance a from the left support and a distance b from the right support. The left reaction is Pb/l. For the left portion of the beam

$$M = \frac{Pbx}{l} \tag{87.1}$$

and to the right of the load

$$M = \frac{Pbx}{l} - P(x - a) \tag{87.2}$$

For all points from $x = 0$ to $x = a$, inclusive,

$$EI\frac{d^2y}{dx^2} = \frac{Pbx}{l} \tag{87.3}$$

$$EI\frac{dy}{dx} = \frac{Pbx^2}{2l} + C_1 \tag{87.5}$$

For all points from $x = a$ to $x = l$, inclusive,

$$EI\frac{d^2y}{dx^2} = \frac{Pbx}{l} - P(x - a) \tag{87.4}$$

$$EI\frac{dy}{dx} = \frac{Pbx^2}{2l} - \frac{P(x - a)^2}{2} + C_3 \tag{87.6}$$

The curve is continuous under the load with no abrupt change of slope. When $x = a$, the value of dy/dx calculated from Eq. (87.5) is the same as when calculated from Eq. (87.6). This makes the first members of the two equations equal and, consequently, the second members are equal when a is substituted for x.

$$\frac{Pba^2}{2l} + C_1 = \frac{Pba^2}{2l} - \frac{P(a - a)^2}{2} + C_3 \tag{87.7}$$

$$C_1 = C_3$$

Substituting C_1 for C_3 in Eq. (87.6) and integrating both equations,

$$EIy = \frac{Pbx^3}{6l} + C_1x + C_2 \tag{87.8}$$

When $x = 0$, $y = 0$; hence

$$C_2 = 0$$

$$EIy = \frac{Pbx^3}{6l} - \frac{P(x - a)^3}{6} + C_1x + C_4 \tag{87.9}$$

When $x = a$, the values of y from Eqs. (87.8) and (87.9) are the same and the second members of these equations are equal, from which

$$0 = C_2 = C_4$$

When $x = l$ in Eq. (87.9), $y = 0$,

$$C_1 = -\frac{Pbl^2}{6l} + \frac{P(l - a)^3}{6l} = -\frac{Pb}{6l}(l^2 - b^2) \tag{87.10}$$

Substituting the value of C_1 from Eq. (87.10) in Eq. (87.8),

$$EIy = \frac{Pbx^3}{6l} - \frac{Pb(l^2 - b^2)x}{6l} \tag{87.11}$$

Substituting C_1 in Eq. (87.5) and equating to zero,

$$x^2 = \frac{l^2 - b^2}{3} = \frac{a(a + 2b)}{3} \tag{87.12}$$

gives the point of maximum deflection, provided b is less than a.

Substituting x from Eq. (87.12) in Eq. (87.11),

$$y_{max} = -\frac{Pb(l^2 - b^2)\sqrt{3(l^2 - b^2)}}{27EIl} = -\frac{Pba(a + 2b)\sqrt{3a(a + 2b)}}{27EIl} \tag{87.13}$$

The deflection under the load is

$$y = -\frac{Pa^2b^2}{3EIl} \tag{87.14}$$

Problem

87-1. A 3- by 2-in. simply supported rectangular beam 10 ft long carries a load of 36 lb 6 ft from the left end. Find the maximum deflection if $E = 1{,}500{,}000$ psi. Find the slope at the left support. Find the slope at the right support and the deflection under the load.
Ans. $y_{max} = -0.4096$ in.; $dy/dx = -0.009677$ at left; $dy/dx = 0.010159$ at right; $y = -0.3981$ in. under load.

88. Deflection Caused by Shear. When a beam is bent, part of the deflection is caused by shear (unless the moment is constant). Previous derivations have neglected the deflection of shear since it is small except in special cases of short deep beams subjected to heavy loads. The distribution of shearing stress in a beam is not uniform, causing the cross sections of the beam (heretofore assumed plane) to become curved sections. However, a solution may be had by considering the shearing strain at the center of gravity of the cross sections of the beam.

$$\frac{dy_s}{dx} = \gamma = \frac{S_s}{G} = \frac{\alpha V}{AG} \tag{88.1}$$

where V/A is the average shearing stress of the cross section and α is a factor used to obtain the maximum shearing stress for this particular cross section. From Eq. (70.10), α is $\frac{3}{2}$ for a rectangular beam, and from Prob. 71-3 it is $\frac{4}{3}$ for a round beam. For I beams a value between 2 and 3 should be used for α. The complete differential equation for deflection is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} + \frac{\alpha\, dV}{AG\, dx} \tag{88.2}$$

The solution of the general case is beyond the scope of this book. However an example will illustrate the method. When a cantilever or a simply supported beam is subjected to a single concentrated load, Eq. (88.1) may be integrated over the length for which V is constant.

$$y_s = \int_0^x \frac{\alpha V\, dx}{AG} = \frac{\alpha V x}{AG} \tag{88.3}$$

Example 1

A 2- by 6-in. rectangular steel beam 10 in. long is used as a cantilever with an 18,000-lb load concentrated on the end. Find the deflection caused by shear and by bending.

$$y_s = \frac{\alpha Vx}{AG} = \frac{3 \times 18{,}000 \times 10}{2 \times 12{,}000{,}000 \times 12} = 0.00187 \text{ in.}$$
$$y = \frac{PL^3}{3EI} = \frac{18{,}000 \times 1{,}000}{3 \times 30{,}000{,}000 \times 36} = 0.00555 \text{ in.}$$

It will be shown in Art. 166 by a method of work and energy that the deflection of a beam of rectangular section when not restrained (as a cantilever is restrained at the wall) is sometimes calculated by using $\alpha = 1.2$. This applies to the case of a simply supported beam with a concentrated load at the middle.

Example 2

A 2- by 4-in. rectangular steel beam 12 in. long is supported at the ends and carries a 24,000-lb load at the middle. Find the deflection caused by shear, using $\alpha = 1.2$, and compare with the deflection of bending.

$$y_s = \frac{1.2Vx}{AG} = \frac{1.2 \times 12{,}000 \times 6}{8 \times 12{,}000{,}000} = 0.0009 \text{ in.}$$
$$y = \frac{PL^3}{48EI} = \frac{24{,}000 \times 1{,}728}{48 \times 30{,}000{,}000 \times 10.67} = 0.0027 \text{ in.}$$

If the load and the depth of the beam are doubled, the deflection caused by shear remains the same while the deflection caused by bending becomes 0.000675.

Example 3

Find the deflection of a 10-in. 25.4-lb steel I beam supported at points 12 in. apart and loaded with 49,600 lb at the middle of the span.

From a study of the distribution of shearing stress on this beam, $\alpha = 2.75$ approximately. Area of web = 3.10 sq in.

$$y_s = \frac{2.75 \times 24{,}800 \times 6}{3.10 \times 12{,}000{,}000} = 0.0110 \text{ in.}$$

The deflection which is due to bending is

$$y = \frac{49{,}600 \times 12^3}{48 \times 30{,}000{,}000 \times 122.1} = 0.00048 \text{ in.}$$

In this extreme case, the deflection which is due to shear is greater than that which is due to bending. If the beam were made twice as long, the bending deflection would be eight times as great, while the shear deflection would be only doubled. For beams of any considerable length relative to their cross section, the deflection which is due to shear may be neglected.

Problems

88-1. A 2- by 3-in. steel beam rests on supports 12 in. apart and carries a load of 12,000 lb midway between the supports. What is the shear deflection and what is the bending deflection? *Ans.* $y_s = 0.0006$ in.; $y = 0.0032$ in.

88-2. Solve Prob. 88-1 if the length of the beam is 24 in. and the load is 6,000 lb.

88-3. Solve Prob. 88-1 if the length of the beam is 60 in. and the load is 2,400 lb.

89. Miscellaneous Problems

89-1. A 6- by 2-in. wood beam 17 ft long is supported at 4 ft from the left end and 3 ft from the right end. It carries a uniform load of 45 lb per ft over the left 4 ft and 80 lb per ft over the right 3 ft with no load over the 10 ft between the supports. Neglect the beam's own weight and find the radius of curvature of the 10-ft span. Find the deflection at the middle of the 10-ft span. Find the maximum bending stress and indicate on your sketch of the beam where it occurs. Find the maximum shearing stress and indicate its location. *Ans.* $y_{max} = -1.62$ in.

89-2. Two 4-in. 1.57-lb aluminum channels are placed back to back and used to replace an I beam 6 ft long supported at the ends. Neglect the weight of the beam and find the center deflection when a 2,000-lb concentrated load is placed at the center of the span. *Ans.* $y_{max} = 0.191$ in.

89-3. The load in Prob. 89-2 is moved from the middle to 2 ft from the right support. Find the center deflection and the maximum deflection.

89-4. An 8-in. 18.4-lb standard steel I beam is supported at the ends on a 5-ft span with the web horizontal. A 2,000-lb load is applied at 2 ft from each end. Consider the weight of the beam and find the maximum deflection by combining several equations.

CHAPTER 10

DEFLECTION BY AREA MOMENTS

90. Method of Area Moments. When a beam is fixed at one or both ends, making the slope zero at these points, the calculation of deflection by area moments[1] has some advantages. Where there is more than one load on a span, the method is better than successive integration with arbitrary constants but is not superior to successive integration between limits.[2] Most calculations of area moments may be made by "graphic integration." This is the determination of the areas of moment diagrams. Since moment diagrams are triangles, parabolas, rectangles, or combinations of these figures, most of these calculations can be made without the use of calculus.

From the equation $M = EI\, d\theta/dl = EI\, d\theta/dx$, when θ is a small angle,

$$d\theta = \frac{M}{EI}\, dx \tag{90.1}$$

$$\theta_B - \theta_A = \frac{1}{EI} \int_A^B M\, dx \tag{90.2}$$

when I is constant.

Since $\int M\, dx$ is the area of the moment diagram, *the difference in slope between two points on a beam of uniform section is the area of the moment diagram between these points divided by EI.* If the moment of inertia varies, the difference of slope is the area of the M/I diagram divided by E.

$$\theta_A + \frac{\text{area of moment diagram between } A \text{ and } B}{EI} = \theta_B \qquad \text{Formula XVIII}$$

In order to obtain correct signs for the slopes, it is important to note that $x = A$ is the lower limit, and $x = B$ is the upper. The moment equations are written, the moment diagram is drawn, and the above

[1] This method was devised by Mohr and independently in America by Prof. Charles E. Greene, who began to teach it in 1873. See the paper by A. E. Greene in the *Michigan Technic* of June, 1910.

[2] Integration between limits is omitted from this book. It was contained in Chap. IX of the 4th edition.

equation is written, *all* in the direction in which x progresses through A and then to B.

In Fig. 138, A_0B_0 represents a portion of a beam. The line $A_1B_1B_0$, which is straight from A_1 to B_1 and curved from B_1 to B_0, represents the same beam after the portion from B_1 to B_0 has been bent by a moment M. If the short portion between B_1 and B_2 is now bent by a moment M, the point A_1 moves to A_2. The angle between the tangent A_1B_1 and the tangent A_2B_2 (or the angle between the normals at B_1 and B_2) is $d\theta$. (It is assumed that the angles are so small that the angle θ is practically equivalent to a tangent of θ, and the points $A_0A_1A_2$ lie on a straight line which is approximately vertical.) The deflection dy_a, from A_1 to A_2, when the infinitesimal length B_1B_2 is bent,

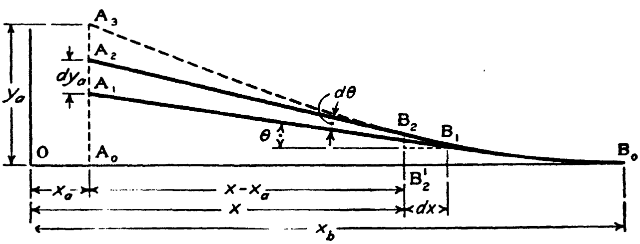


FIG. 138. Derivation of area moments.

is $A_2B_2\,d\theta$ (or $A_1B_1\,d\theta$, since B_1B_2 is infinitesimal). Since θ is small, A_2B_2 is practically equivalent to its horizontal projection A_0B_2', the length of which is $x - x_a$.

$$dy_a = (x - x_a)\,d\theta \tag{90.3}$$

$$dy_a = \frac{M}{EI}(x - x_a)\,dx \tag{90.4}$$

The total deflection of the point A is

$$y_A = \int_A^B \frac{M}{EI}(x - x_a)\,dx \tag{90.5}$$

For a uniform beam,

$$y_A = \frac{1}{EI}\int_A^B Mx\,dx \tag{90.6}$$

when the origin of coordinates is taken at the point A_0.

Since $M\,dx$ is an element of the moment diagram, $(x - x_a)\,M\,dx$ is the moment of this element with respect to the line $A_0A_1A_2$. The integrals give the moment with respect to A of the entire moment diagram from A to B_0. This is called the *area-moments method.*

$$y_A = \frac{\text{moment (about } A\text{) of area of moment diagram between } A \text{ and } B}{EI}$$

Formula XIX

The deflection y_A obtained will be measured from a tangent line drawn on the beam at B.

For a concentrated load or reaction, the moment diagram is a triangle; for a uniformly distributed load it is a parabola. Since the area and location of the center of gravity of these figures are known, the moment is usually computed geometrically without integration. This method of replacing $\int xf(x)\,dx$ by the moment of the area $\int f(x)\,dx$ is a form of graphic integration.

$$\int xf(x)\,dx = \bar{x}\int f(x)\,dx$$

if $\int f(x)\,dx$ may be represented by a plane figure, of which the area and the location of the center of gravity are known, $\bar{x}\int f(x)\,dx$ may be evaluated arithmetically. When $\int f(x)\,dx$ cannot be represented by a simple plane figure, it is best to integrate.

91. Cantilever Loaded at Free End. Figure 139 shows a cantilever with a load P on the free end. The figure shows also the elastic line

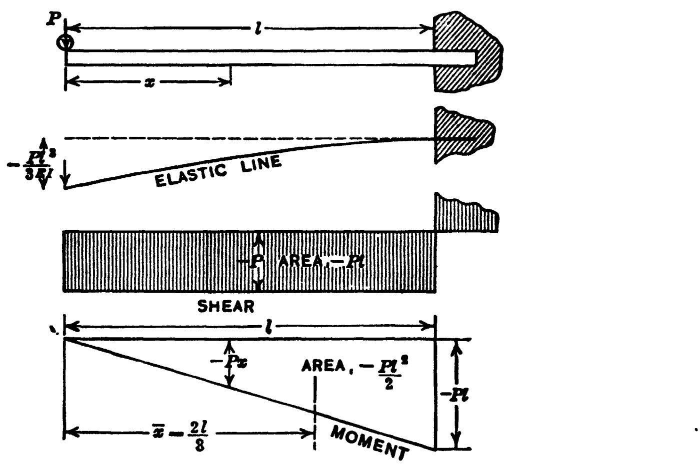


FIG. 139. Area moments for cantilever.

with deflections magnified, the shear diagram, and the moment diagram. The moment diagram is a negative triangle. The maximum ordinate is $-Pl$, which is the area of the shear diagram. The area of the moment triangle is $-Pl^2/2$ and its center of gravity is $2l/3$ from the left end. The deflection of the left end from the horizontal tangent

at the right end is given by

$$EIy_{max} = -\frac{Pl^2}{2} \times \frac{2l}{3} = -\frac{Pl^3}{3} \tag{91.1}$$

$$y_{max} = -\frac{Pl^3}{3EI} \tag{91.2}$$

To find the slope at the free end A,

$$\theta_A + \left(-\frac{Pl^2}{2EI}\right) = \theta_B \tag{91.3}$$

At the fixed end,

$$\theta_B = 0$$

$$\theta_A = \frac{Pl^2}{2EI}$$

The equation of the elastic line may be found by finding the deflection at some point, say A, at a distance x from the end of the beam.

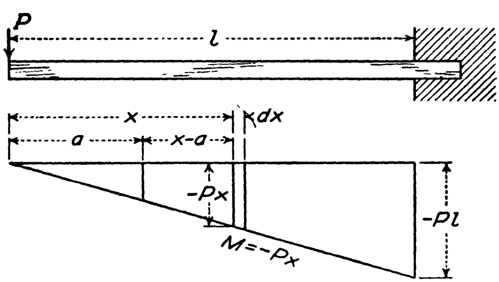


FIG. 140. Moment diagram for equation of elastic line.

To illustrate how this may be done for any loading, three methods are shown:

1. *By integrating the diagram,* or the *integral of $Mx\,dx$ method.* In Fig. 140 the coordinate is expressed by a instead of x since x is used for the element of the diagram. Equation (90.5) is written

$$EIy_A = -P\int_a^l x(x-a)\,dx \tag{91.4}$$

$$EIy_A = -P\left[\frac{x^3}{3} - \frac{ax^2}{2}\right]_a^l = -\frac{P}{6}(2l^3 - 3al^2 + a^3) \tag{91.5}$$

When a is replaced by x in the parentheses of Eq. (91.5), the result is Eq. (80.8).

2. *By graphic integration* when the moment diagram is a simple geometric figure for which the area and center of gravity are known.

In Fig. 141 find the deflection at a distance x from the free end. This point is labeled B, since the student should not be dependent on a particular notation but should understand the method. The trapezoid to the right of the point B may be broken up into the rectangle of base $l - x$ and altitude $-Px$, and the lower triangle of base $l - x$ and altitude $-P(l - x)$.

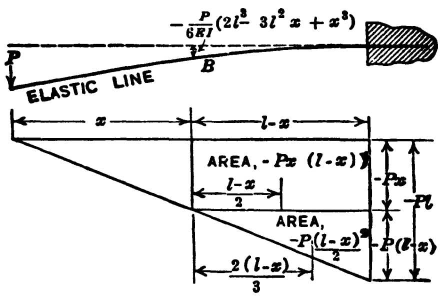


FIG. 141. Area moments for any point of cantilever.

The moment arm of the rectangle is $\frac{1}{2}(l - x)$ and that of the triangle is $2(l - x)/3$.

$$\text{Moment of rectangle} = -Px(l - x) \times \frac{l - x}{2} = -\frac{Px}{2}(l - x)^2 \tag{91.6}$$

$$\text{Moment of triangle} = -\frac{P(l - x)^2}{2} \times \frac{2(l - x)}{3} = -\frac{P(l - x)}{3}(l - x)^2 \tag{91.7}$$

$$\text{Total moment} = EIy = -\frac{Pl^3}{3} + \frac{Pl^2x}{2} - \frac{Px^3}{6} \tag{91.8}$$

$$y = -\frac{P}{6EI}(2l^3 - 3l^2x + x^3) \tag{91.9}$$

3. *By graphic integration, with this difference.* Instead of the *sum* of the moments of the triangle and the rectangle to the right of B, the *difference* of the moments of the entire triangle and the small triangle to the left of B may be used. The area of the large triangle is $-Pl^2/2$, and its center of gravity is $(2l/3) - x$ to the *right* of B. The area of the small triangle is $-Px^2/2$ and its center of gravity is $x/3$ to the left of B.

$$EIy = -\frac{Pl^2}{2} \times \left(\frac{2l}{3} - x\right) - \left(-\frac{Px^2}{2}\right) \times \left(-\frac{x}{3}\right) \tag{91.10}$$

Note the origin of the three negative signs on the last term: first, the term is subtracted because too much area was included in the large triangle; second, the area is negative; and third, the moment arm is negative, being measured to the left, while the positive direction of x is to the right in this case.

$$y = -\frac{P}{6EI}(2l^3 - 3l^2x + x^3) \tag{91.11}$$

Problems

(Do not use formulas. Work each problem from its moment diagram.)

91-1. Find the slope of the elastic line at a distance x from the free end by means of the area of the trapezoid.

91-2. A beam of length l carries a load P on the free end. Find the deflection at $0.4l$ from the free end by the moment of the moment diagram. Divide trapezoid into a triangle and a rectangle. Also divide the trapezoid into two triangles.

91-3. A 4- by 6-in. beam 15 ft long is fixed at the right end. It carries a load on the free end which makes the maximum stress 1,080 psi. Find the deflection at the middle if $E = 1{,}200{,}000$ psi. Solve by area moments from the numerical moment diagram.

Ans. $y = -\dfrac{17{,}496{,}000 + 69{,}984{,}000}{72 \times 1{,}200{,}000} = -1.0125$ in.

91-4. A cantilever 5 ft long is deflected 2.4 in. at the free end. How much is it deflected 20 in. from the free end? Solve by moments of moment diagram without use of formulas.

91-5. A 6- by 2-in. rectangular timber beam 5 ft long is fixed at the right end and carries 120 lb on the free end. Find the deflection and the slope at a point 15 in. from the free end. Write the numerical values on the moment diagram. *Ans.* $y = -1.14$ in.

91-6. In Fig. 142 the cantilever is fixed at the left end and x is measured from an origin at the wall. Find the deflection and the slope at the free end.

91-7. From the diagram of Fig. 142, find the equation of the elastic line, measuring x from the wall as shown.

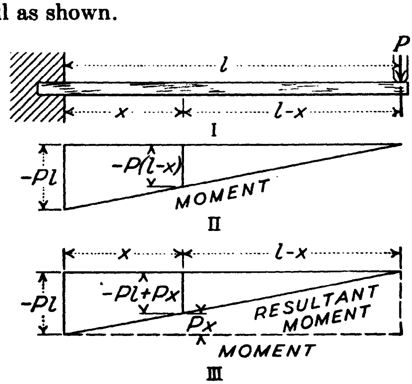


FIG. 142. Cantilever fixed at left end.

92. Cantilever Loaded at Any Point. Figure 143 shows a cantilever of length l, which carries a load P at a distance a from the free end. The area of the moment triangle is $-P(l-a)^2/2$. The moment arm for finding the deflection at the free end is $a + [2(l-a)/3]$, which reduces to $(2l + a)/3$.

$$EIy = -\frac{P(l-a)^2}{2} \times \frac{2l+a}{3} = -\frac{P}{6}(2l^3 - 3l^2a + a^3) \quad (92.1)$$

$$y_{\max} = -\frac{P}{6EI}(2l^3 - 3l^2a + a^3) \quad (92.2)$$

The result is identical with Eq. (91.5). This identity is an illustration of Maxwell's theorem.

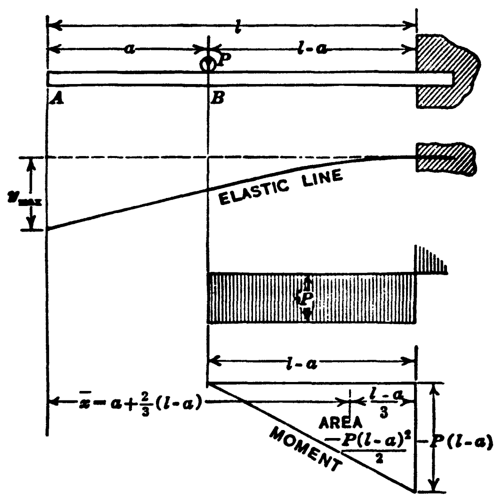


FIG. 143. Cantilever with load at any point.

Problems

92-1. A 6- by 4-in. horizontal timber cantilever 9 ft long carries a 400-lb load concentrated at 6 ft from the wall. Find the deflection and slope at the free end. Do all work on the moment diagram without the use of formulas.

Ans. -2.27 in.

92-2. A cantilever beam 9 ft long is fixed at the right end and carries 100 lb on the free end, and 200 lb at 3 ft from the free end. If $EI = 14{,}400{,}000$ lb-in.^{2}, find the deflection and slope under the 200-lb load. Solve by drawing two separate triangular diagrams for the loads and combining the answers.

Ans. $\theta = 0.072$ rad.

92-3. A 4- by 4-in. cantilever 15 ft long carries a load of 40 lb 5 ft from the free end and a load of 60 lb 10 ft from the free end. Find the deflection at the free end if E is 1,350,000 psi. Solve from the two moment triangles.

Ans. $y_{\max} = -2$ in

92-4. A 6- by 2-in. timber beam 80 in. long is fixed at the right end. Originally the beam was not quite horizontal but was inclined slightly upward to the left, so that when a 200-lb load was applied at 60 in. from the wall, a point B 30 in. from the wall had a horizontal tangent to the beam. Find the original slope of the unloaded beam. Find also the elevations of the free end, the load, and the point B, relative to the fixed end of the beam.
HINT: Measure deflections from a tangent line at B.

Ans. $\theta = -0.05625$ rad $= 3°13'$; $y_{end} = 0$.

92-5. A cantilever beam 80 in. long is fixed at the left end and carries a 300-lb load at 20 in. from the free end. Find the deflection and slope at 40 in. from the wall, if $EI = 10{,}000{,}000$ lb-in.²

93. Cantilever Uniformly Loaded. Figure 144 shows a cantilever with uniformly distributed load of w per unit length. The shear dia-

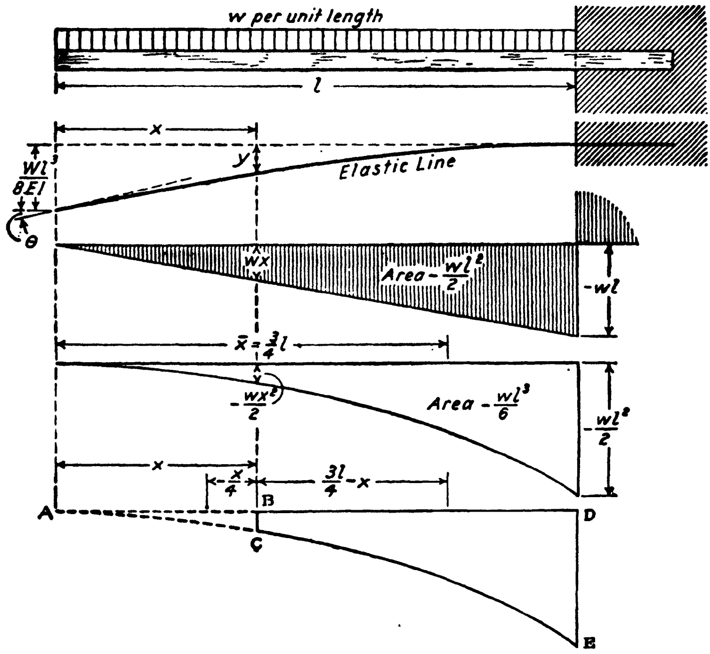


FIG. 144. Cantilever with uniformly distributed load.

gram is a triangle of maximum altitude $-wl$ and area $-wl^2/2$. The moment diagram is a parabola, the equation of which is $M = -wx^2/2$. The maximum ordinate of the moment diagram is $-wl^2/2$. The area of a parabola which is convex toward the base is one-third the product of the base by the maximum height.

$$\text{Area of moment diagram} = -\frac{wl^2}{2} \times \frac{l}{3} = -\frac{wl^3}{6} \qquad (93.1)$$

Since the slope at the fixed end is zero, the slope at the free end is given by

$$\theta_A - \frac{wl^3}{6EI} = \theta_B = 0$$

$$\theta_A = \frac{wl^3}{6EI} \tag{93.2}$$

The distance of the center of gravity of this parabola from the vertex is $3l/4$, which corresponds to the center of gravity of a pyramid or cone.

To find the deflection at the free end,

$$EIy_{\max} = -\frac{wl^3}{6} \times \frac{3l}{4} = -\frac{wl^4}{8} \tag{93.3}$$

$$y_{\max} = -\frac{wl^4}{8EI} = -\frac{Wl^3}{8EI} \tag{93.4}$$

in which $W = wl =$ total distributed load.

1. *To derive the equation of the elastic line by integrating the moment diagram*, moments are taken about any point at a distance a from the free end. The vertical element of area of the moment diagram (not shown in Fig. 144) is $-wx^2/2$ high and dx wide. Its moment arm with respect to the point at a distance a from the free end is $x - a$.

$$EIy = -\frac{w}{2}\int_a^l x^2(x - a)\,dx = -\frac{w}{2}\left[\frac{x^4}{4} - \frac{ax^3}{3}\right]_a^l \tag{93.5}$$

$$EIy = -\frac{w}{24}(3l^4 - 4l^3a + a^4) \tag{93.6}$$

Substituting x for a,

$$y = -\frac{w}{24EI}(3l^4 - 4l^3x + x^4) \tag{93.7}$$

2. *To find the deflection at any point by graphic integration*, the moment of the parabola to the left of B (Fig. 144) is subtracted from the moment of the entire parabola ADE. The area of the entire parabola is $-wl^3/6$, and its moment arm with respect to BC is $\frac{3l}{4} - x$. The area of the small parabola ABC is $-wx^3/6$, and its moment arm, measured from BC toward the *left*, is $-x/4$.[1]

[1] The positive direction of moment arm is from the point about which moment is taken toward the point of tangency of the line from which deflections are measured. The direction from B toward D of Fig. 144 is positive.

$$EIy = \left(-\frac{wl^3}{6}\right)\left(\frac{3l}{4} - x\right) - \left(-\frac{wx^3}{6}\right)\left(-\frac{x}{4}\right) \tag{93.8}$$

which gives Eq. (93.7).

3. *To find the deflection at any point by graphic integration using the general moment equation*, if the horizontal dimension of the element of Fig. 109 is x, it can be shown that the moment about the right side of

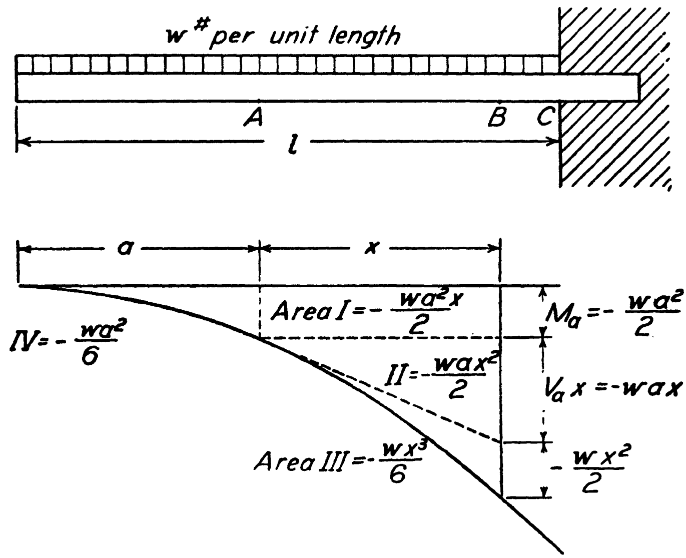


FIG. 145.

the element can be expressed in terms of the moment and shear on the left side and the intervening loads.

$$M_x = M + Vx - \frac{wx^2}{2} \tag{93.9}$$

Let A in Fig. 145 be an origin where the moment and shear are known.

$$M_A = \frac{-wa^2}{2} \qquad V_A = -wa$$

$$M_B = -\frac{wa^2}{2} - wax - \frac{wx^2}{2} \tag{93.10}$$

Note that the slope at any point on the moment curve is $dM/dx = V$, and hence the area of the moment diagram between A and B can be divided into a rectangle, a triangle whose hypotenuse is tangent to the curve, and a parabolic area. Taking moments about A of the area

between A and B gives

$$EIy = -\frac{wa^2x}{2} \times \frac{x}{2} - \frac{wax^2}{2} \times \frac{2x}{3} - \frac{wx^3}{6} \times \frac{3x}{4} \tag{93.11}$$

$$y = -\frac{wx^2}{24EI}(6a^2 + 8ax + 3x^2) \tag{93.12}$$

This does not appear to check Eq. (93.7), but it must be remembered that x is a variable distance between A and B. If $(l - x)$ is substituted for x, and x for a, the result will check Eq. (93.7).

Problems

93-1. Show that Eq. (93.12) checks Eq. (93.7) by making the required substitutions.

93-2. A cantilever 30 in. long carries 4 lb per in. If $EI = 10{,}000{,}000$ lb-in.², find the deflection at the free end and at 12 in. from the free end. Solve by means of the numerical areas of the moment diagram without using formulas.
Ans. -0.0405 in.; -0.0192 in.

93-3. A 6- by 4-in. timber cantilever 10 ft long carries a distributed load of 36 lb per ft. If $E = 1{,}500{,}000$ psi, find the deflection at the free end and find the maximum stress. *Ans.* $y = -1.62$ in.; $S = 1{,}350$ psi.

93-4. In Prob. 93-3, find the slope at the left end and at the middle from the moment diagram. *Ans.* $dy/dx = 0.018$; 0.01575.

93-5. A 3-in. 6-lb. standard steel channel with its web horizontal, projects 15 ft horizontally from a vertical wall. How much is the free end deflected by its own weight? What is the slope at the free end? Solve from the moment diagrams without using formulas. Find the maximum bending stress.
Ans. $y = -1.04$ in.; $S = 5{,}790$ psi.

93-6. Find the deflection at $0.4l$ from the free end of a uniformly loaded cantilever by subtracting the moment of the small moment parabola of length $0.4l$ from the moment of the large moment parabola of length l.

93-7. Solve Prob. 93-6 by means of the rectangle, triangle, and parabola which make up the moment diagram from $x = 0.4l$ to the fixed end.

93-8. In Fig. 146, find the slope and deflection at a distance x from the fixed end.

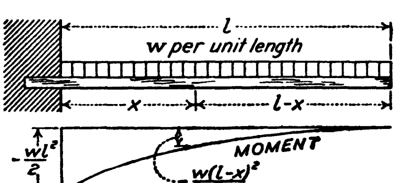


FIG. 146. Uniformly loaded cantilever fixed at left end.

93-9. From an expansion of the general moment equation the moment at x in Fig. 146 is

$$M = -\frac{wl^2}{2} + wlx - \frac{wx^2}{2}$$

which may be represented by a negative rectangle, a positive triangle, and a negative parabola. Using these as the moment diagram instead of the one shown, find the deflection at the free end and at x.

94. Cantilever Partly Loaded. Figure 147 shows a cantilever which carries a distributed load over a portion of its length adjacent to the fixed end, while the remainder, of length a, is not loaded. To

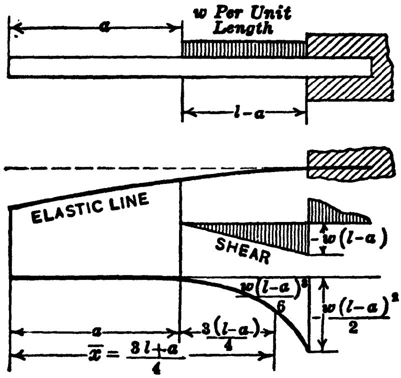


FIG. 147. Cantilever with distributed load over part of length.

find the deflection at the free end, the area of the moment diagram is $-w(l - a)^3/6$, and the moment arm with respect to the end is

$$a + \frac{3}{4}(l - a) = \frac{3l + a}{4} \tag{94.1}$$

$$EIy_{max} = -\frac{w(l - a)^3}{6} \times \frac{3l + a}{4} \tag{94.2}$$

$$y_{max} = -\frac{w}{24EI}(3l^4 - 8l^3a + 6l^2a^2 - a^4) \tag{94.3}$$

The slope of the straight-line portion of length a is $w(l - a)^3/6EI$.

Example 1

A cantilever of length l carries a load of w per unit length over 0.6 of its length adjoining the fixed end. Find the deflection at the free end, at $0.2l$ from the free end, and at $0.4l$ from the free end by the moment diagram without the use of formulas. Make a sketch of the beam, the moment diagram, and the elastic line similar to Fig. 147 for $a = 0.4l$.

$$\text{Maximum moment} = -0.18wl^2$$
$$\text{Area of moment diagram} = -0.036wl^3$$
$$y_{max} = -\frac{0.036wl^3(0.4l + 0.45l)}{EI} = -\frac{0.0306wl^4}{EI}$$

At $0.2l$,

$$EIy = -0.036wl^3 \times 0.65l = -0.0234wl^4$$

At $0.4l$,

$$EIy = -0.036wl^3 \times 0.45l = -0.0162wl^4$$

Example 2

A cantilever of length l carries a load of w per unit length over 0.6 the length adjacent to the free end and no load over the remainder. Find the deflection at the free end and at $0.6l$ from the free end.

Figure 148 shows the loaded beam; Fig. 148,II is the moment diagram. The negative parabola from A to B has a maximum moment of $-0.18wl^2$ at $0.6l$ from the free end. The remainder of the diagram is a straight line which is the moment

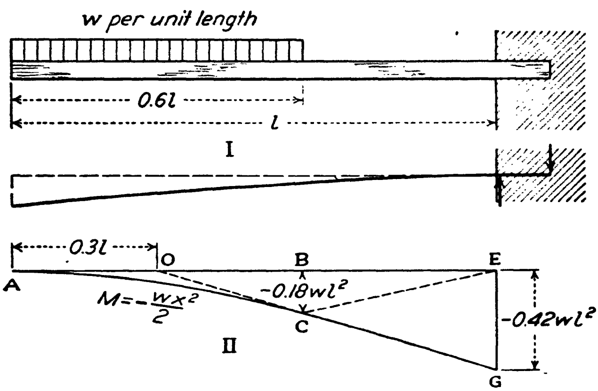


FIG. 148. Partly loaded cantilever.

of the load $0.6wl$ concentrated at $0.3l$ from the end of the beam. The maximum moment at the wall is $-0.6wl \times 0.7l = -0.42wl^2$. The trapezoid $BEGC$ is divided into two triangles. To find the deflection at the free end,

Parabola	$-0.036wl^3 \times 0.45l$	=	$-0.0162wl^4$
Triangle BCE	$-0.036wl^3 \times {}^{22}\!/_{30}l$	=	$-0.0264wl^4$
Triangle CEG	$-0.084wl^3 \times {}^{26}\!/_{30}l$	=	$-0.0728wl^4$
	$EIy_{\max}$	=	$-0.1154wl^4$

At $0.6l$ from the end,

$$EIy = -0.036wl^3 \times \frac{4l}{30} - 0.084wl^3 \times \frac{8l}{30} = -0.0272wl^4$$

Problems

94-1. A beam of length l, fixed at the right end, carries a uniformly distributed load over $0.8l$ adjacent to the fixed end. Find the deflection at the free end, at $0.2l$ from the free end, and at $0.4l$ from the free end. Solve each directly from the moment of the moment diagram.

94-2. A cantilever beam 2 by 6 in. is $7\frac{1}{2}$ ft long and fixed at the right end. The beam is considered weightless and carries 120 lb per ft over the 5 ft adjacent to the wall, and 100 lb concentrated at the free end. If $E = 1{,}000{,}000$ psi, find the deflection at the free end and at 30 in. from the free end. Since the principle of superposition is valid for deflections, draw separate moment

diagrams for the loads and add the two solutions without using any derived formulas. *Ans.* $y_{max} = -1.425$ in.

94-3. A 6- by 4-in. timber beam $7\frac{1}{2}$ ft long is fixed at the right end and carries a uniform load of 60 lb per ft over the 5 ft adjacent to the free end, and no uniform load on the rest of the beam. It also carries a 120-lb load concentrated at 50 in. from the free end. Find the maximum deflection and the maximum bending stress. *Ans.* $y_{max} = -1.211$ in.; $S = 1{,}425$ psi.

95. Simply Supported Beam, Load at Middle. A beam which is supported at the ends and loaded at the middle is shown in Fig. 149.

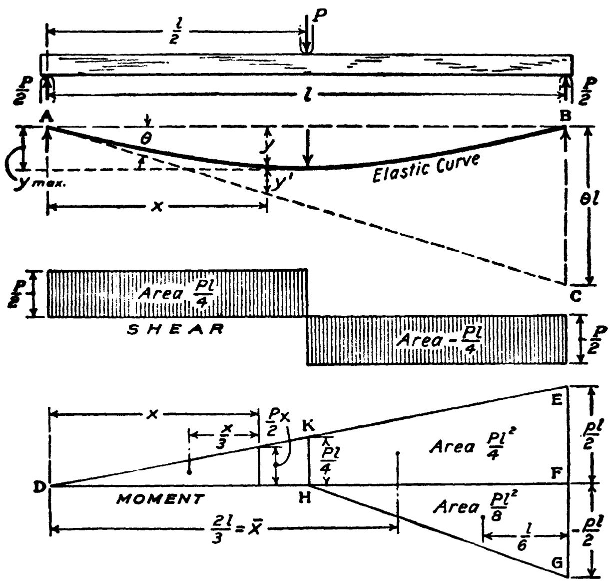


FIG. 149. Simply supported beam, load at middle.

The moment diagram is composed of two triangles, the ordinate of DEF being the moment of the left reaction, and the ordinate of HFG being the moment of the load. By symmetry the maximum deflection occurs under the load. Since deflections in the area-moment method are measured from the tangent line which is drawn on the beam at some point, it is better to consider the beam equivalent to two cantilevers, held horizontal at the middle and bent upward by the end reactions. The deflection of A upward from a horizontal tangent at P is the moment of the area DHK about A divided by EI.

$$EIy = \left(\frac{1}{2}\right)\left(\frac{Pl}{4}\right)\left(\frac{l}{2}\right)\left(\frac{2}{3}\right)\left(\frac{l}{2}\right) = +\frac{Pl^3}{48} \qquad (95.1)$$

This is positive, being measured upward from the tangent. The deflection of the middle below the level of the supports is

$$y_{max} = -\frac{Pl^3}{48EI} \tag{95.2}$$

Example 1

Find the slope at the left end of the beam.

$$\theta_A + \frac{\text{Area } DHK}{EI} = \theta_M = 0 \tag{95.3}$$

$$\theta_A = -\frac{Pl^2}{16EI} \tag{95.4}$$

Example 2

Derive the equation of the elastic line for the left half of the beam, using Eq. (95.4).

$$y = \theta_A x - y'$$

$$EIy = -\frac{Pl^2x}{16} - \left(\frac{Px}{2}\right)\left(\frac{x}{2}\right)\left(-\frac{x}{3}\right) \tag{95.5}$$

The arm is negative, as explained in the footnote on page 184.

Problems

95-1. A 6- by 6-in. beam is supported at points 80 in. apart and carries a center load of 4,000 lb. If $E = 1{,}500{,}000$ psi find the center deflection, neglecting the weight of the beam.

95-2. A 20-in. I beam rests on supports 25 ft apart and carries a 42,000-lb load at the middle such that the maximum bending stress is 20,000 psi. Select the beam and then find its maximum deflection, neglecting the beam's own weight. Solve from the moment diagram without formulas.

Ans. −0.493 in.

95-3. From the moment diagram, show that the equation of the elastic line from $x = l/2$ to $x = l$ is

$$y = -\frac{P}{48EI}(-l^3 + 9l^2x - 12lx^2 + 4x^3)$$

96. Simply Supported Beam, Uniformly Loaded. The moment at a distance x from the left support is $(wlx/2) - (wx^2/2)$. The diagrams for these terms are shown separately at the bottom of Fig. 150. Any small triangle cut off by an ordinate at x from the left vertex has a maximum ordinate of $wlx/2$, an area of $wlx^2/4$, and its center of gravity at $2x/3$ from the origin of coordinates. The moment diagram for the uniformly distributed load is the negative parabola $M = -wx^2/2$. The area of this parabola is $-wx^3/6$ and its center of gravity is at $3x/4$ from the origin.

The combined moment diagram is *EFD* of Fig. 150. The maximum ordinate at the middle is $wl^2/8$. Since the area of a parabola concave toward the base is *two-thirds* the product of the base by the altitude, its area is $wl^3/12$. Its center of gravity is at $\bar{x} = l/2$. One-half of this parabola has an area of $wl^3/24$, and its center of gravity is at $5l/16$ from the origin. When part of the triangle or parabola of the *separate* diagrams is cut off by an ordinate at a distance x from the origin, this

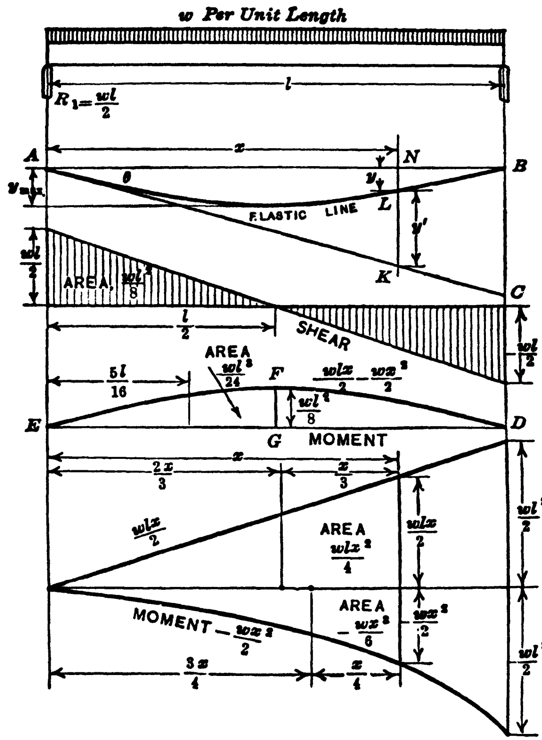


FIG. 150. Beam supported at ends, distributed load.

portion is another smaller triangle or parabola for which the area and the location of the center of gravity are known. For the *combined* diagrams, only the entire figure or one-half of it can be used conveniently. For this reason, separate diagrams are preferable when the equation of the elastic line is required. On the other hand, in the solution of indeterminate beams, which employ the moment of the entire span to find the reactions and moments at the supports, the combined diagram has many advantages.

From symmetry, it is evident that the beam supported at the ends

and uniformly loaded (Fig. 150) is horizontal at the middle. It is possible, therefore, to find the deflection of any point from the tangent at the middle, and then find the equation of the elastic line by subtracting from this the deflection of the end from the tangent at the middle.

The deflection of the left end from the tangent at the middle is

$$EIy_A = \left(\frac{wl}{2}\right)\left(\frac{l}{2}\right)\left(\frac{l}{4}\right)\left(\frac{2l}{6}\right) - \left(\frac{wl^2}{8}\right)\left(\frac{l}{6}\right)\left(\frac{3l}{8}\right) \tag{96.1}$$

The center deflection will be equal to y_A but it will be negative.

$$y_{max} = -\frac{5wl^4}{384EI} = -\frac{5Wl^3}{384EI} \tag{96.2}$$

To find the slope at the left end,

$$\theta_A + \frac{wl^3}{16EI} - \frac{wl^3}{48EI} = \theta_M = 0 \tag{96.3}$$

$$\theta_A = -\frac{wl^3}{24EI} \tag{96.4}$$

Problems

96-1. Find the deflection at one-fourth the length from the left end and at three-fourths the length from the left end from the value of θ and the moment of the moment diagrams without using any other equations from the text.

96-2. From the slope at the left end and the area of the moment diagram, find the slope at the right end.

96-3. From the slope of the left end and the deflection of the beam as measured from this slope, find the equation of the elastic line of the beam.

Ans. $y = -\frac{wx}{24EI}(l^3 - 2lx^2 + x^3)$.

96-4. A 6- by 12-in. wood beam 15 ft long between supports carries a total uniform load of 6,000 lb. Find the maximum deflection and the maximum bending stress. *Ans.* $y_{max} = -0.439$; $S = 938$ psi.

96-5. A 5- by 3- by ⅜-in. 4.14-lb aluminum tee is supported with the flange horizontal on supports 30 ft apart. Find the maximum deflection caused by its own weight. Find the maximum bending stress.

Ans. $y_{max} = -3.00$ in.; $S = 5{,}270$ psi.

96-6. A 12- by 8-in. wide-flange 40-lb beam 20 ft long is supported at the ends and carries 400 lb per ft including the weight of the beam, and a 4,300-lb concentrated load at the middle of the span. Find the deflection under the load working directly from the moment diagrams. Find the maximum bending stress. *Ans.* $y_{max} = -0.288$ in.

97. Simply Supported Beam, Load at Any Point. The moment diagram consists of a positive triangle of base l and a negative triangle

of base b, as shown in Fig. 151. It is necessary to use a general method which is applicable to all beams supported at the ends. This consists in (1) getting the expression for the deflection of the right support from a tangent at the left end; (2) finding the slope at the left end; (3) finding the deflection upward of the elastic line from the tangent at the left end, and finally (4) combining the deflections algebraically to get the position of the beam with respect to the horizontal line joining the supports. The solution follows.

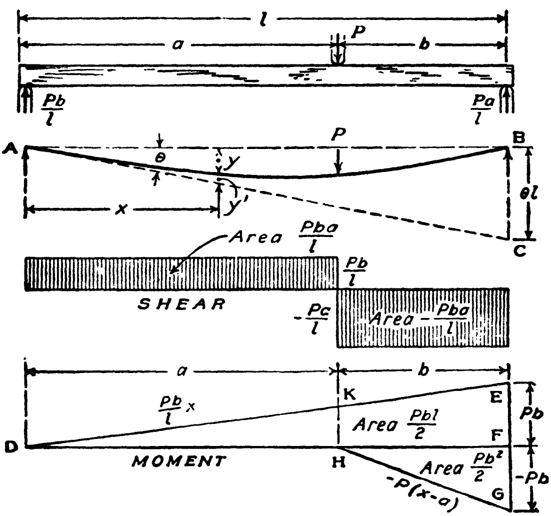


FIG. 151. Simply supported beam, load at any point.

To find the deflection *upward* of the right end of the *elastic line* from the *straight line* which is tangent at its left end, the moment of the separate moment diagrams about the right end gives

$$EI \times CB = \frac{Pbl}{2} \times \frac{l}{3} - \frac{Pb^2}{2} \times \frac{b}{3} = \frac{Pb}{6}(l^2 - b^2) \qquad (97.1)$$

$$CB = \frac{Pb}{6EI}(l^2 - b^2) \qquad (97.2)$$

The slope at the left end is found by dividing CB by the length. However, θ must be considered negative when measured downward from the horizontal line AB.

$$\theta = \tan\theta = -\frac{CB}{l} = -\frac{Pb}{6EIl}(l^2 - b^2) \qquad (97.3)$$

The deflection y at any point is the algebraic sum of the deflection of the tangent at that point (which is negative) and deflection y' of the elastic line from the tangent (which is positive).

Under the load,

$$EIy' = \frac{Pba^2}{2l} \times \frac{a}{3} = \frac{Pba^3}{6l} \tag{97.4}$$

$$y = a\theta + y' = -\frac{Pba}{6EIl}(l^2 - b^2) + \frac{Pba^3}{6EIl} \tag{97.5}$$

$$y = -\frac{Pba}{6EIl}(l^2 - b^2 - a^2) = -\frac{Pa^2b^2}{3EIl} \tag{97.6}$$

At a distance x from the left support (if x is less than a),

$$EIy' = \frac{Pbx^2}{2l} \times \frac{x}{3} = \frac{Pbx^3}{6l} \tag{97.7}$$

$$y = -\frac{Pbx}{6EIl}(l^2 - b^2) + \frac{Pbx^3}{6EIl} \tag{97.8}$$

$$y = -\frac{Pbx}{6EIl}(l^2 - b^2 - x^2) \tag{97.9}$$

Equation (97.9) applies from $x = 0$ to $x = a$. Beyond the load the moment of a small moment triangle, of length $x - a$ and altitude $-P(x - a)$, must be divided by EI and the quotient added to Eq. (97.9).

At the point of maximum deflection, the slope θ_M is zero.

$$\theta + \frac{\text{area}}{EI} = \theta_M = 0$$

$$-\frac{Pb}{6EIl}(l^2 - b^2) + \frac{Pbx^2}{2EIl} = 0 \tag{97.10}$$

$$x^2 = \frac{l^2 - b^2}{3} \qquad x = \sqrt{\frac{l^2 - b^2}{3}} \tag{97.11}$$

Equation (97.11) gives the abscissa of the point of maximum deflection, provided b is less than one-half the length. If b is greater than one-half the length, the point of maximum deflection falls to the right of the load, and another term $P(x - a)^2/2EI$ must be added to the slope equation.

The point of maximum deflection always lies between the load and the middle and is much closer to the middle than to the load.

When x from Eq. (97.11) is substituted in Eq. (97.8),

$$y_{max} = -\frac{Pb(l^2 - b^2)\sqrt{3(l^2 - b^2)}}{27EIl} = -\frac{Pba(a + 2b)\sqrt{3a(a + 2b)}}{27EIl} \tag{97.12}$$

Problems

(Formulas should not be memorized. Problems should be worked from the moment diagrams.)

97-1. Write the equation for the elastic line for the portion of the beam to the right of the load. Measure x as in Fig. 152.

$$Ans.\ y = -\frac{Pbx}{6EIl}(l^2 - b^2 - x^2) - \frac{P(x-a)^3}{6EI}.$$

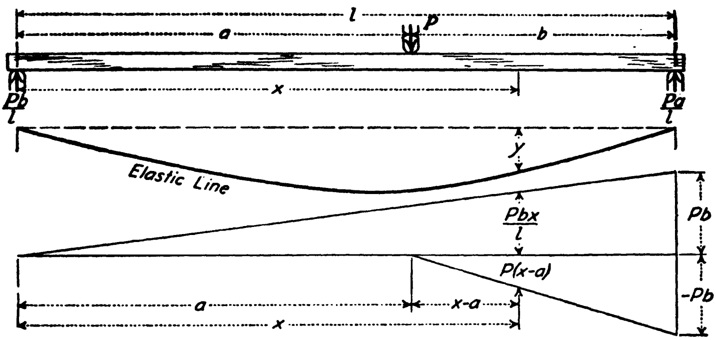


FIG. 152. Simply supported beam with load at any point.

97-2. A beam is supported at points 100 in. apart and loaded 40 in. from the right support. Find the deflection under the load. *Ans.* $y = -19{,}200P/EI$.

97-3. Find the location of the maximum deflection of the beam of Prob. 97-2, and calculate maximum deflection.

Ans. $x^2 = 2{,}800$, $x = 52.915$ in.; $y_{max} = -19{,}755P/EI$.

97-4. Find the deflection at the middle of the beam of Prob. 97-2.

Ans. $y = -19{,}667P/EI$.

97-5. A 6- by 2-in. rectangular wood beam $7\frac{1}{2}$ ft long is supported at the ends. A 300-lb load is applied at 5 ft from the right end. Neglecting the weight of the beam, find the slope at both ends of the beam. Find the deflection under the load and the maximum deflection.

Ans. $\theta_R = 0.025$; $y_{max} = -0.816$ in.

97-6. A 5- by 6-in. timber beam rests on supports 100 in. apart and carries a load of 1,200 lb 30 in. from the right support. If $E = 1{,}000{,}000$ psi, find the deflection at 20, 40, 50, 60, and 70 in. from the left support.

Ans. 0.116 in.; 0.200 in.; 0.220 in.; 0.220 in.; 0.196 in.

98. Beam of Constant Moment. If M is a constant moment, the moment diagram is a rectangle of height M. By symmetry the beam is horizontal at the middle. Using Formula XVIII and calling the slope at the left end θ_1,

$$\theta_1 + \frac{Ml}{2EI} = \theta_M = 0 \tag{98.1}$$

$$EI\theta_1 = -\frac{Ml}{2} \tag{98.2}$$

$$EIy = -\frac{Mlx}{2} + Mx \times \frac{x}{2}$$

$$EIy = -\frac{Mx(l-x)}{2} \tag{98.3}$$

$$y_{max} = -\frac{Ml^2}{8EI} \tag{98.4}$$

Constant moment may be applied by clamps at the ends of the beam. The portion of the beam between the supports in Figs. 129 and 136 will be subjected to constant moment (neglecting the weight of the beam). In Fig. 153 the portion of the beam between the equal loads has constant moment.

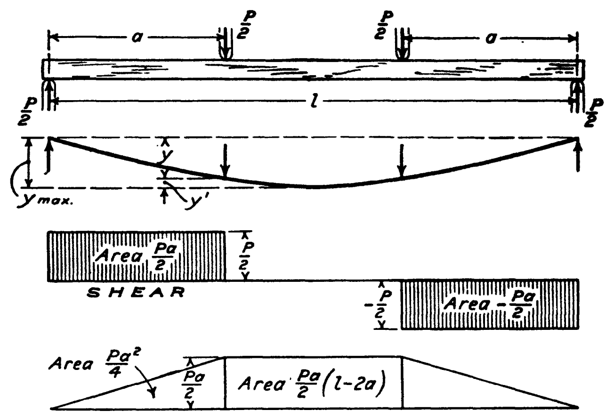


FIG. 153. Moment constant between loads.

Problems

98-1. For the loads of Fig. 153, find the slope at the left end of the beam.

Ans. $\theta_1 = -\frac{Pa(l-a)}{4EI}$.

98-2. In Fig. 153, find the deflection under the left load.

Ans. $y = -\frac{Pa^2}{12EI}(3l - 4a)$.

98-3. In Fig. 153, find the maximum deflection.

Ans. $y_{max} = -\frac{Pa}{48EI}(3l^2 - 4a^2)$.

98-4. Find the deflection under the load in Fig. 153 and at the middle when the loads are at the third points.

Ans. $y_a = -\frac{5Pl^3}{324EI}$; $y_{max} = -\frac{23Pl^3}{1{,}296EI}$.

98-5. A beam 100 in. long carries 80 lb 30 in. from the left support and 60 lb 40 in. from the right support. Find the deflection at the 80-lb load. Solve from a diagram similar to Fig. 153.

Ans. $EIy = -81{,}200 \times 30 + 36{,}000 \times 10 = -2{,}076{,}000$.

98-6. Find the deflection under the 60-lb load for Prob. 98-5.

Ans. $y = -2{,}352{,}000/EI$.

98-7. What would be the deflection of the beam of Prob. 98-5 if the two loads were placed together at the middle?

98-8. Find the position of maximum deflection in Prob. 98-5.

98-9. An 8-in. 23-lb standard I beam rests on supports 20 ft apart. It carries a load of 4,800 lb 5 ft from the left end, and an equal load 5 ft from the right end. Find the deflection of the middle below the loads.

Ans. $y = -0.269$.

98-10. Find the total deflection of the middle below the supports in Prob. 98-9.

Ans. $y_{max} = -0.987$ in.

99. Successive Differentiation of Equation of Elastic Line. It has been shown in preceding articles that the derivative of the deflection is the slope, the derivative of the slope is the moment, and the derivative of the moment is the shear. To show some of these relations a series of equations may be written as follows:

$$EI\frac{d^4y}{dx^4} = EI\frac{d^3\theta}{dx^3} = \frac{d^2M}{dx^2} = \frac{dV}{dx} = w \tag{99.1}$$

in which w is the load per unit length.

$$EI\frac{d^3y}{dx^3} = EI\frac{d^2\theta}{dx^2} = \frac{dM}{dx} = V = \text{shear equation} \tag{99.2}$$

$$EI\frac{d^2y}{dx^2} = EI\frac{d\theta}{dx} = M = \text{moment equation} \tag{99.3}$$

$$\frac{dy}{dx} = \theta = \text{slope equation} \tag{99.4}$$

$$y = \text{equation of elastic line} \tag{99.5}$$

In order to differentiate or integrate an expression, the function must be continuous. A concentrated load is, apparently, not continuous. It is customary to represent a concentrated load by a line. In reality, a concentrated load or reaction is distributed over an area. The concentrated load of Fig. 154 is represented on the load diagram by a rectangle. Using a rectangle, the shear diagram would be the diagonal broken line of the figure. It is customary to use a vertical line on the shear diagram in line with the resultant "concentrated" load. At any rate, the shear after passing the load is equal to the total load or the area of the actual load diagram, whatever its form may be. Figure 154 shows a uniformly distributed load, for which the shear diagram $B'C$ is an inclined straight line. A uniformly increasing load is shown on the right end of the beam of Fig. 154. If ub is the load per unit

length at a distance b from the beginning, the equation of the load diagram is $-ux'$. The additional shear caused by this load at a distance x' from the beginning is the area of the load triangle of length x', which is $-ux'^2/2$. For most deflection problems, the moment equation may be written from the loads and reactions. For some problems,

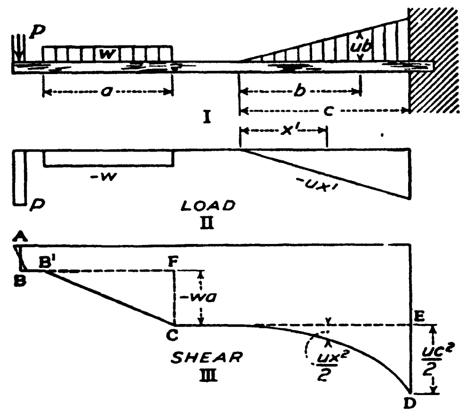


FIG. 154. Types of loading.

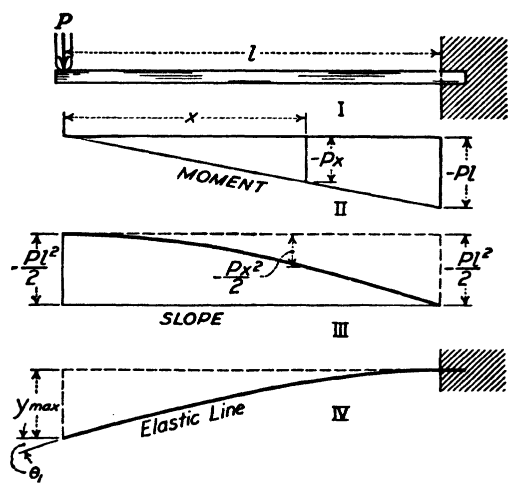


FIG. 155. Moment and slope of cantilever.

it is convenient to begin with the shear, and for others it is necessary to integrate from the load.

In Fig. 154, Eqs. (99.1) and (99.2) are shown graphically, while Fig. 155 shows the last three equations graphically for another condition of loading.

Figure 156 shows a cantilever with uniformly increasing load.

$$V = -\int ux\,dx = -\left[\frac{ux^2}{2}\right]_0^x = -\frac{ux^2}{2} \tag{99.6}$$

which might have been calculated by graphic integration from the area of the load triangle.

$$M = -\int \frac{ux^2}{2}\,dx = -\left[\frac{ux^3}{6}\right]_0^x = -\frac{ux^3}{6} \tag{99.7}$$

which might have been calculated by graphic integration from the area of the load parabola.

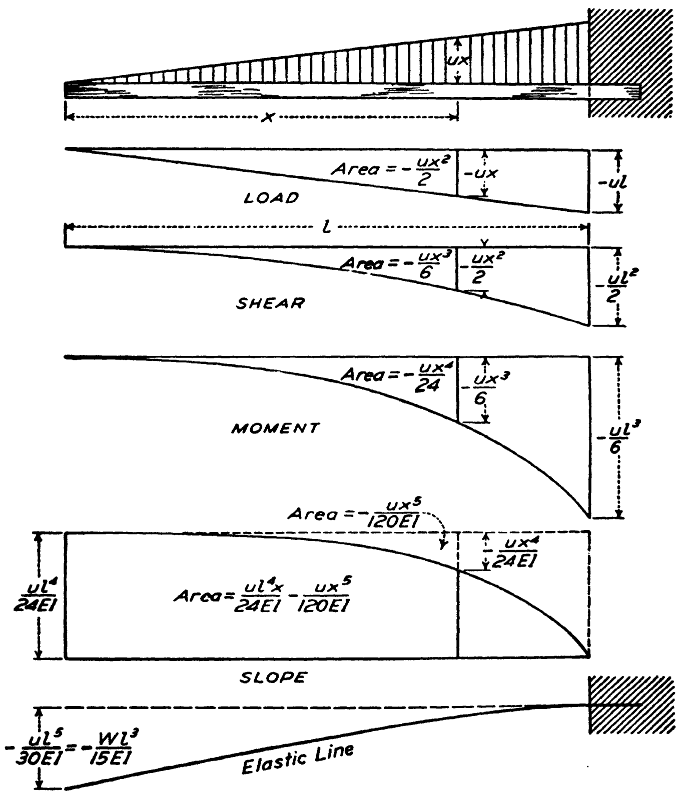


FIG. 156. Uniformly increasing load.

Note that each equation, (99.1) through (99.5), is illustrated by a diagram, and that the load w is not constant but must be expressed as an equation also. By the usual methods the deflection may be found. (The center of gravity of the area of the moment diagram is four-fifths of the length from the end.)

$$y_{max} = -\frac{ul^5}{30EI} = -\frac{Wl^3}{15EI} \tag{99.8}$$

in which $W = ul^2/2$, the total load on the beam.

Problems

99-1. A cantilever of length l carries a load which increases uniformly from the free end to the fixed end. Find the deflection at the middle.
Ans. $y = -49ul^5/3{,}840EI = -49Wl^3/1{,}920EI$.

99-2. By use of the moment diagram, write the equation of the elastic line for the beam of Fig. 156.
Ans. $EIy = -\frac{ul^5}{30} + \frac{ul^4x}{24} - \frac{ux^5}{120}$.

99-3. A beam 40 in. long is fixed at the right end and carries a load which increases from the free end to the wall where the intensity is 30 lb per in. If $EI = 10{,}000{,}000$ lb-in.², find the deflection at the free end. Work from the moment diagram. *Ans.* -0.256 in.

99-4. A cantilever of length l carries a load which increases uniformly from the fixed end to the free end. Find the maximum deflection.
Ans. $y_{max} = -11Wl^3/60EI$.

100. Miscellaneous Problems

100-1. A 6- by 10-in. beam of timber, 20 ft long between supports, carries a uniformly distributed load which makes the unit stress the maximum allowable in bending. Find the deflection at the middle and at 5 ft from each support. Find the slope at the left support. *Ans.* $y_{max} = -1.2$ in.

100-2. A 12- by 10-in. wide-flange 53-lb beam 15 ft long is supported at the ends 15 ft apart and carries a uniform load of 3,000 lb per ft including the beam. By area moments find the maximum deflection. Find the maximum bending stress. *Ans.* -0.267 in.; 14,300 psi.

100-3. Find the deflection of the beam of Prob. 100-2 at 5 ft from the left support.

100-4. A 20-in. 75-lb standard steel I beam with web horizontal is supported at points 25 ft apart. Find the total uniformly distributed load which makes the maximum bending stress 18,000 psi. Find the maximum deflection when this load is applied.
Ans. $W = 4{,}512$ lb; $y_{max} = -1.756$ in.

100-5. Solve Prob. 100-4 if the uniformly distributed load is replaced by a load concentrated at the middle of the span.

100-6. A weightless 4- by 6-in. rectangular beam is 15 ft long and supported at 50 in. from each end. A concentrated 1,200-lb load is applied at 80 in. from the right end. If $E = 1{,}000{,}000$ psi, find how much each end is elevated above the line of the supports. *Ans.* $y_L = 0.286$ in.

100-7. In Prob. 100-6, find the deflection under the load. Find from the diagram the position of maximum deflection. *Ans.* $x = 42.8$ in. from left support.

100-8. A cantilever beam carries a uniformly distributed load over $0.3l$ adjacent to the free end and $0.2l$ adjacent to the fixed end. Find the deflection at the free end. *Ans.* $y = -0.07911wl^4/EI$.

100-9. A 4- by 6-in. cantilever 10 ft long carries a uniformly distributed load which makes the maximum stress 1,200 psi. Find the deflection at the end and at 40 in. from the free end if $E = 1{,}500{,}000$ psi.

100-10. At what point on a uniformly loaded cantilever is the slope one-half the slope at the free end? *Ans.* $x = 0.5^{1/3}l = 0.794l$.

100-11. A 4- by 6-in. timber cantilever is 10 ft long and carries a load at 30 in. from the free end which makes the maximum unit stress 1,080 psi. Find the deflection at the middle, under the load, 10 in. from the free end, and at the free end, if E = 1,500,000 psi. Fix the beam at the left end as shown in Fig. 157,I and use the moment diagram for the beam with the portion to the right of the section taken as the free body as in Fig. 157,II.
Ans. −0.336 in.; −0.648 in.; −0.864 in.; −0.972 in.

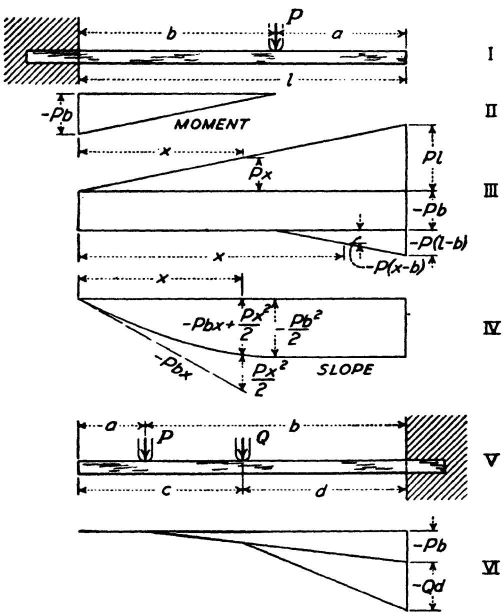


FIG. 157. Cantilever variously loaded.

100-12. Solve for the deflection under the load and at the free end for the beam of Prob. 100-11. Use Fig. 157,III, which gives the components of the diagram with the portion to the left of the section taken as the free body. By the general moment equation,

$$M = -Pb + Px - P(x - b)$$

which are represented by a negative rectangle, a positive triangle, and a negative triangle.

100-13. A 4-in. 7.7-lb standard steel I beam 10 ft long is used as a cantilever fixed at the right end. It carries two concentrated loads: 200 lb at 4 ft from the free end, and 300 lb at $2\frac{1}{2}$ ft from the fixed end. Figure 157,VI shows the moment diagram drawn for convenient computing. Find the deflection at the free end. *Ans.* −0.362 in.

100-14. In Prob. 100-13, what per cent error is made in neglecting the deflection caused by the weight of the beam? Find the maximum bending stress.
Ans. 26.5 per cent; 9,300 psi.

100-15. A beam is supported at the ends and carries a uniform load over a portion adjacent to the right end as shown in Fig. 158. Find the slope of the beam over the left support. Find the deflection at the left end of the uniform load.

$$Ans.\ \theta_1 = -\frac{wb^2}{24EI}(2l^2 - b^2).$$

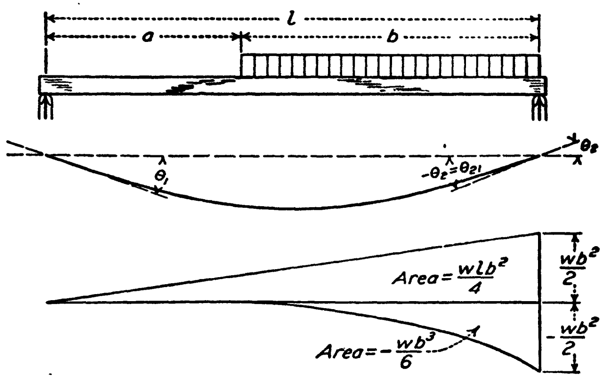


FIG. 158. Load uniformly distributed over part of span.

100-16. A weightless beam is supported at the ends, 90 in. apart. It carries a uniformly distributed load of 2 lb per in. over the 30 in. adjacent to the right support and no load over the remainder of the beam. If the constant EI = 10,000,000 lb-in.², find the slope of the beam at the left end and the maximum deflection. Work directly from the moment diagram without formulas.

$Ans.\ \theta_L = -0.001275;\ x = 50.5$ in.; $y_{max} = -0.0429$ in.

100-17. A weightless beam is supported at the ends and carries a load which increases uniformly from left support to right support as shown in Fig. 159. Find the slope at the left support and the position of maximum deflection. Find also the location of the dangerous section.

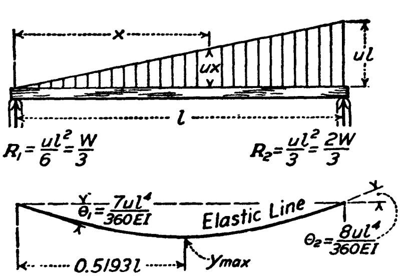


FIG. 159. Uniformly increasing load.

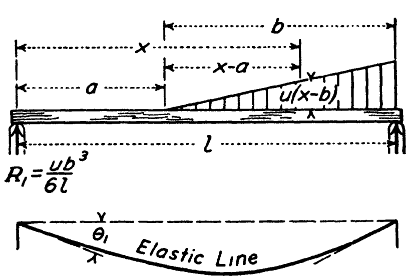


FIG. 160. Uniformly increasing load over part of span.

100-18. Find the slope of the beam at the left end for the beam shown in Fig. 160. Find the deflection at distance a from the left support.

$$Ans.\ EI\theta_1 = \frac{ub^3l}{6} - \frac{ub^5}{20l}.$$

CHAPTER 11

INDETERMINATE BEAMS

101. Determinate and Indeterminate Beams. A beam or structure is statically determinate when it is possible to compute external reactions and resisting moments by elementary statics without reference to deformations or elastic constants. A beam on two supports is *statically determinate.* A beam on three supports in the same plane is *statically indeterminate.* If the beam is *assumed* to be *absolutely rigid* and the supports *perfectly inelastic,* any infinitesimal displacement of one support will throw all the load on two supports. For an actual beam, which is always flexible, it is possible to calculate the reaction on any number of supports by means of deflection equations.

Much space is allotted in this book to the deflection of beams. Since stress is more important, it might seem that deflection has received undue prominence. However, for indeterminate beams, external reactions and moments, which are necessary for the calculation of stress, may be found only by means of deflection formulas.

102. The General Moment Equation. In Art. 93 the general moment equation was used to break the moment curve down into

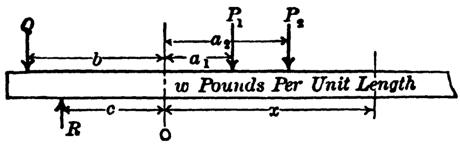


FIG. 161. General case of loading.

convenient parts. It is often desirable to be able to write the moment equation with any point as the origin. Figure 161 represents a beam of indefinite extent with the origin of coordinates on a vertical line through O. To the right of the origin, at distance a_1, a_2, etc., there are concentrated loads P_1, P_2, etc. There is also a uniformly distributed load of w per unit length. There may be any number of vertical loads and reactions to the left of the origin, but all the vertical loads may be replaced by their resultant Q at some definite distance b from the origin, and all the vertical reactions by a single reaction R at a distance c from the origin. Writing the moment with respect

to a section at a distance x from the origin,

$$M = R(c + x) - Q(b + x) - P_1(x - a_1) - P_2(x - a_2) - \frac{wx^2}{2} \quad (102.1)$$

$$M = Rc - Qb + (R - Q)x - P_1(x - a_1) - P_2(x - a_2) - \frac{wx^2}{2} \quad (102.2)$$

$Rc - Qb$ is the moment at the origin, which may be represented by M_0, and $R - Q$ is the shear at the origin, which may be represented by V_0.

$$M = M_0 + V_0x - P_1(x - a_1) - P_2(x - a_2) - \frac{wx^2}{2} \quad (102.3)$$

$$M = M_0 + V_0x - \Sigma P(x - a) - \frac{wx^2}{2} \qquad \text{Formula XX}$$

in which $\Sigma P(x - a)$ represents the sum of the moments of all the concentrated loads between the origin and the section considered. When any point on a beam is taken as the origin of coordinates, the moment at any section at a distance x to the right of the origin is the moment at the origin, plus the shear at the origin multiplied by the distance of the section from the origin, plus the moment with respect to the section of each load and reaction between the origin and the section.

103. Types of Moment Diagrams. The ease of manipulating the moment diagram depends usually on the type of diagram which is used to express the moment. Figure 162 shows one span of a continuous beam. M_a is the moment about R_a of all the forces (not shown) to the left of R_a. If the shear diagram were drawn, V_{ab} could be read from the diagram at a point just to the right of R_a. If the origin for x is taken at V_{ab}, Formula XX may be written

$$M = M_a + V_{ab}\,x - \frac{wx^2}{2} \quad (103.1)$$

To find M_b at the second support,

$$M_b = M_a + V_{ab}\,l - \frac{wl^2}{2} \quad (103.2)$$

$$V_{ab} = \frac{M_b - M_a}{l} + \frac{wl}{2} \quad (103.3)$$

When this value of V is substituted in Eq. (103.1),

$$M = M_a + \frac{M_b - M_a}{l} x + \frac{wlx}{2} - \frac{wx^2}{2} \tag{103.4}$$

The terms $(wlx/2) - (wx^2/2)$ together give the moment of a simply supported beam which is uniformly loaded. These may be called the *simple-support* moment expression. The curve ACB should be compared with Fig. 107. Ordinates above the line AB are positive, while those below will usually be negative. The trapezoid may be called the *end-moment trapezoid.* Figure 162,I is easy to manipulate if the entire

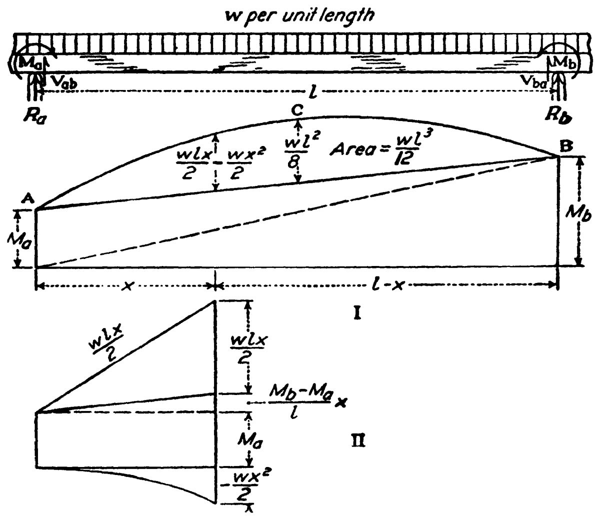


FIG. 162. Separate and combined moment diagrams.

span is under consideration, but if the portion x is needed, the diagram can be broken down by the general moment equation into the parts shown in II. This diagram should be extended to the right support if the entire span is to be worked by this method. Graphically $wlx/2$ is represented by a positive triangle, and $-wx^2/2$ by a negative parabola.

Figure 162,I and II gives two ways of drawing the moment diagram. Previously the combined diagram was drawn as in Fig. 108, but it is not best suited to finding deflections by area moments. The student should recognize the different types of moment diagrams which are used and should be able to draw and utilize the kind best suited to the purpose.

104. Uniformly Loaded Span Fixed at One End. Figure 163 shows a beam which is fixed at the right end and supported at the left end.

The beam is supposed to be originally horizontal, weightless, and to touch the support under no load.

To find the unknown reaction at the support of the beam of Fig. 163 by area moments, the deflection at the support from the tangent at the

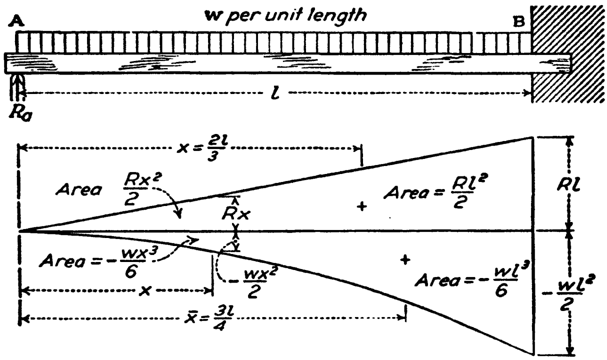


FIG. 163. Separate diagrams for distributed load.

fixed end is calculated from the reaction triangle and the load parabola of that figure.

$$EIy_a = 0 = \frac{Rl^2}{2} \times \frac{2l}{3} - \frac{wl^3}{6} \times \frac{3l}{4} \tag{104.1}$$

$$\frac{Rl^2}{3} = \frac{wl^3}{8} \qquad R = \frac{3wl}{8} \tag{104.2}$$

Problems

104-1. Calculate M_b from the end reaction and the load.

104-2. Show that the beam may be regarded as a cantilever which is bent downward by the distributed load of w per unit length and bent upward by the concentrated reaction at the free end. Since these deflections are equal and opposite,

$$\frac{Rl^3}{3EI} - \frac{wl^4}{8EI} = 0$$

104-3. A 6- by 2-in. plank is fixed in a vertical wall at the right end and projects to a support which is 10 ft from the wall. The plank carries a uniformly distributed load of 36 lb per ft. Find the reaction at the support. With this reaction known, calculate the moment at the right end. Find the unit stress at the wall.

Ans. R = 135 lb; M = −5,400 in.-lb; S = 1,350 psi.

104-4. Construct the shear diagram for the beam of Prob. 104-3 to the scale of 1 in. equals 2 ft of length and 100 lb of shear. What is the shear at the fixed end? *Ans.* V = −225 lb.

104-5. Find the dangerous section between the support and the fixed end. What is the moment at this dangerous section? Find the position of contraflexure for the beam of Prob. 104-3. *Ans.* M = 3,037.5 in.-lb; x = 90 in.

104-6. Find the maximum deflection of the beam of Prob. 104-3.
HINT: Locate the position of C at a distance x from A. The area of the moment diagram between C and B will be zero. Why?

$$\textit{Ans.}\ 162{,}000 \times 60 - \frac{135x^2}{2} - 216{,}000 \times 40 + \frac{3x^3}{6} = 0.$$

One root of this equation is $x = 120$. Why?

104-7. A 5-in. 14.75-lb I beam projects 10 ft from a vertical wall and rests on a fixed support at the left end. What is the additional reaction of the support when a uniformly distributed load of 6,000 lb is placed on the beam? What is the maximum unit stress at the two dangerous sections and at the middle? What is the shear next to the fixed end?

Ans. $R = 2{,}250$ lb; $S = 8{,}437$, 15,000, and 7,500 psi; $V = -3{,}750$ lb.

104-8. If the support at the end of the beam of Prob. 104-7 settles 0.8 in. when the 6,000-lb load is applied, find the reaction at the support. Find the maximum bending stress in the beam. Find the position of contraflexure.

Ans. $R = 1{,}625$ lb.

104-9. A 4- by 6-in. timber beam 10 ft long is supported at the left end and fixed at the right end in a horizontal position. When a uniform load of 48 lb per ft is applied, find the reaction at the left end, the maximum bending stress, and the maximum shearing stress in the beam.

104-10. Solve Prob. 104-9 if the support settles ½ in. when the load is applied.

104-11. In Fig. 164 the beam is fixed at the left end and supported at the right. The simple-support and end-moment moment diagrams are shown in II,

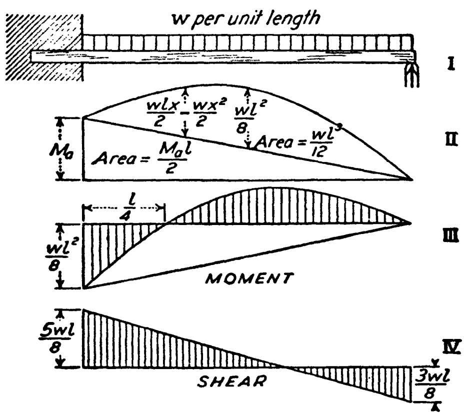


FIG. 164. Combined diagrams for distributed load.

while III gives the combined diagram. Using II, find the moment at the fixed end by means of an equation for the deflection of the supported end.

$$\textit{Ans.}\ EIy = 0 = \frac{M_a l}{2} \times \frac{2l}{3} + \frac{wl^3}{12} \times \frac{l}{2}$$

M_a was assumed positive and came out negative. What is the sign of M_a?

104-12. A 4-in. 3.03-lb standard aluminum I beam 20 ft long is supported at the ends and at the middle as shown in Fig. 165. The beam carries 320 lb

per ft. By area moments find the three reactions if all are at the same horizontal level. Find the maximum bending stress.
HINT: Because of symmetry, either half of the beam may be solved.
Ans. $R_1 = 1{,}200$ lb; $R_2 = 4{,}000$ lb.

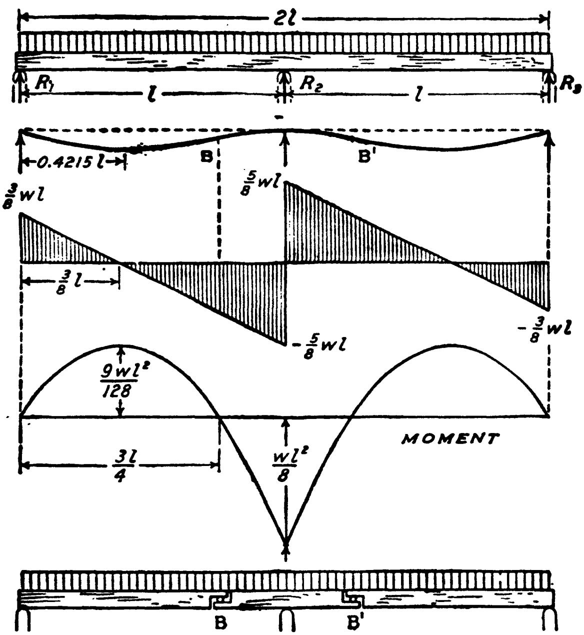


FIG. 165. Beam with three supports.

104-13. The center support of Prob. 104-12 settles 2 in. Find the reactions and the maximum bending stress. *Ans.* $S = 12{,}000$ psi.

105. Span Fixed at One End, Load Concentrated. Figure 166 shows a beam which is fixed at the right end, supported at the left end, and subjected to a load P at a distance a from the supported end and at a distance b from the fixed end.

Figure 166,III shows the positive reaction triangle and the negative load triangle. To find the deflection of the left end from the tangent at the fixed end,

$$EIy = 0 = \frac{Rl^2}{2} \times \frac{2l}{3} - \frac{Pb^2}{2}\left(l - \frac{b}{3}\right) \tag{105.1}$$

$$\frac{Rl^3}{3} = \frac{Pb^2}{2}\left(l - \frac{b}{3}\right) \tag{105.2}$$

$$R = \frac{Pb^2}{2l^3}(3l - b) \tag{105.3}$$

$$R = \frac{P(l - a)^2}{2l^3}(2l + a) \tag{105.4}$$

$$R = \frac{P}{2}(l - k)^2(2 + k) = \frac{P}{2}(2 - 3k + k^3) \tag{105.5}$$

in which $k = a/l$.

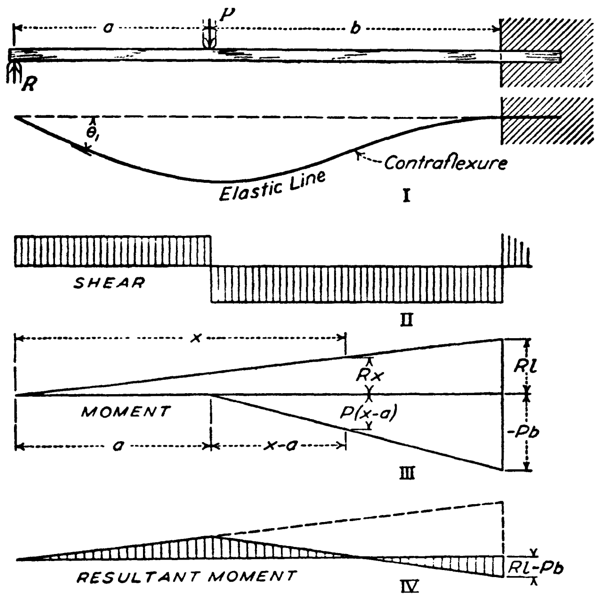


FIG. 166. Separate and combined diagrams for concentrated load.

Problems

105-1. Find the expression for the slope at the left end of the beam in Fig. 166.

Ans. $EI\theta_1 + \dfrac{Rl^2}{2} - \dfrac{Pb^2}{2} = 0$; $EI\theta_1 = -\dfrac{Pb^2a}{4l}$

105-2. A 3- by 4-in. timber beam is supported at the left end and fixed 100 in. from the left end. It carries a load of 400 lb 40 in. from the support. Find the unit stress in the outer fibers under the load and at the fixed end. Work directly from the diagrams without the use of formulas.

Ans. 864 psi; 840 psi.

105-3. Solve Prob. 105-2 if the load is 40 in. from the fixed end of the span.

105-4. In Prob. 105-3. find the position of maximum deflection by working from the diagrams. Find the maximum deflection.

Ans. $x = 48$ in.; $y_{\max} = -0.160$ in., found by taking the moment about the position of maximum deflection of the area of the moment diagram between that point and the fixed end. There are two other ways of finding $y_{\max}$. What are they? Check the answer by one of them.

105-5. A 4-in. 7.7-lb standard I beam, 13 ft 4 in. long, is supported at the ends and at the middle. A load of 3,072 lb is placed 30 in. from the left support

and an equal load is placed 30 in. from the right support. Find the reaction of each support and the stress over the middle support caused by the concentrated loads. *Ans.* 1,425 lb; 3,294 lb; and 1,425 lb; 13,200 psi.

105-6. If the middle support of the beam of Prob. 105-5 is lowered 0.12 in., and $E = 30{,}000{,}000$ psi, what is the reaction of each support?

105-7. In Fig. 167, the beam has been reversed and the moment diagram drawn using the general moment equation

$$M = M_a - \frac{M_a x}{l} + \frac{Pax}{l} - P(x - b)$$

M_a is drawn positive although it turns out to be negative. The first three terms of the moment expression apply to the entire span; the last term applies from the load to the right end.

Solve for M_a, using the general moment diagram shown.

Ans. $M_a = \frac{Pa}{2l^2}(l^2 - a^2)$.

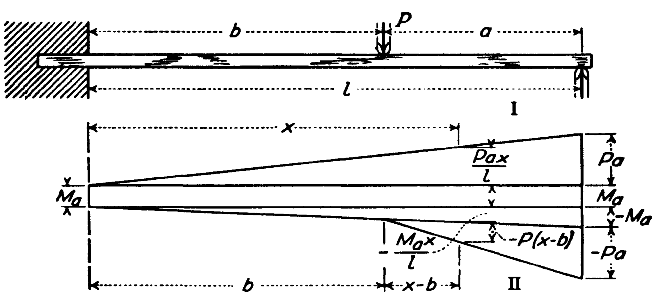


FIG. 167. General moment diagram for beam fixed at left end.

106. Uniformly Loaded Span, Fixed at Both Ends.

From the symmetry of the loading, the shear at each end must be $wl/2$. In Fig. 168 the height of the rectangle is the moment at the left end, and the ordinate to the triangle is the moment caused by the shear at the

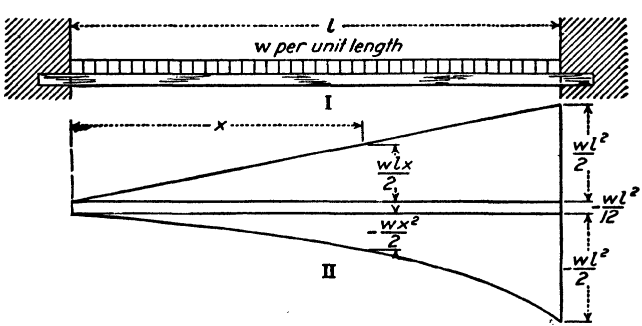


FIG. 168. Diagrams for general moment equation.

left end. Since the slope of the beam is horizontal at both ends, the total area of the moment diagram is zero.

$$M_a l + \frac{wl^2}{2} \times \frac{l}{2} - \frac{wl^2}{2} \times \frac{l}{3} = 0 \tag{106.1}$$

$$M_a = M_b = -\frac{wl^2}{12} \tag{106.2}$$

Problems

106-1. Find the moment at the middle of a uniformly loaded beam which is fixed at both ends. *Ans.* $M = wl^2/24 = Wl/24$.

106-2. Derive the equation of the elastic line for the beam of Fig. 168.

$$Ans.\ EIy = -\frac{wl^2x}{12} \times \frac{x}{2} + \frac{wlx^2}{4} \times \frac{x}{3} - \frac{wx^3}{6} \times \frac{x}{4}$$

$$y = -\frac{wx^2}{24EI}(l - x)^2$$

106-3. Find the end moment for Fig. 168 for the condition that the left end has zero deflection from the tangent line at the right end.

106-4. Find the end moment by means of Fig. 169.

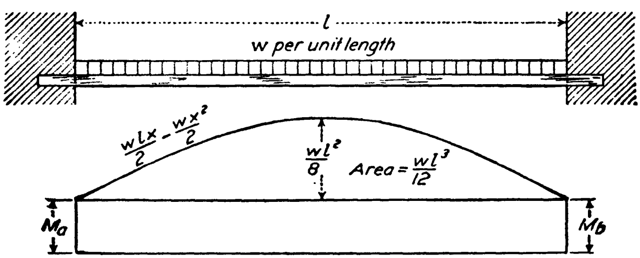


FIG. 169. End-moment and simple-support diagrams.

106-5. Find the position of contraflexure. Use Fig. 168.

106-6. A 6- by 8-in. beam is 20 ft long between fixed ends. It carries a load of 160 lb per ft. What is the maximum unit stress? What would be the maximum unit stress if the beam were simply supported at the ends? Find the maximum deflection. *Ans.* 1,000 psi; 1,500 psi.

When a uniformly loaded span is absolutely fixed at the end, the moment at the ends is $-wl^2/12$ and the maximum positive moment at the middle is $wl^2/24$. The combined moment diagram is shown in Fig. 170,IV. This is not a convenient type for use with area moments. A uniformly loaded beam which is simply supported has the maximum moment of $wl^2/8$. For continuous reinforced-concrete beams, it is customary to regard the ends as partially fixed and to use $-wl^2/10$ as the maximum moment at the ends.

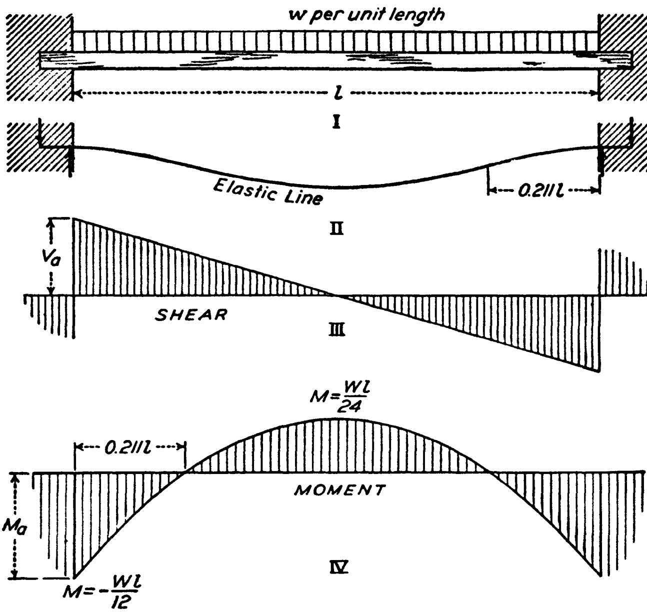


FIG. 170. Shear and resultant moment diagrams.

Problems

106-7. The ends of a uniformly loaded beam are attached to columns which bend sufficiently to make the moment at each end equal to $-wl^2/10$ when the load is applied. Calculate the moment at the middle of the beam and locate the points of inflection.

Ans. $M = wl^2/40$; $x = \frac{5 \pm \sqrt{5}}{10} l = 0.2764l$ or $0.7236l$.

106-8. Two 6- by 3½- by ½-in. aluminum angles, 5.46 lb per ft each, are clamped with the 6-in. legs together to form a T-shaped section and used as a beam with the ends fixed 10 ft apart. When a uniform load of 300 lb per ft including the dead weight is applied, find, by using the diagrams, the end moments, the maximum bending stress, and the center deflection.

Ans. $y_{max} = -0.155$ in.

107. Span Fixed at Both Ends, Load Concentrated. For a beam fixed at both ends with a load concentrated, the slope is zero at both ends, and the area of the resultant moment diagram is zero.

$$EI\theta = 0 = M_a l + \frac{V_a l^2}{2} - \frac{Pb^2}{2} \tag{107.1}$$

The deflection of the right end from the tangent at the left end is zero.

$$EIy = M_a l \times \frac{l}{2} + \frac{V_a l^2}{2} \times \frac{l}{3} - \frac{Pb^2}{2} \times \frac{b}{3} = 0 \tag{107.2}$$

$$M_a = -\frac{Pab^2}{l^2} \tag{107.3}$$

Problems

107-1. The moment at the right end of the beam in Fig. 171 is the total altitude of the moment diagram. Solve for the moment from the equations above.
Ans. $M_b = -Pa^2b/l^2$.

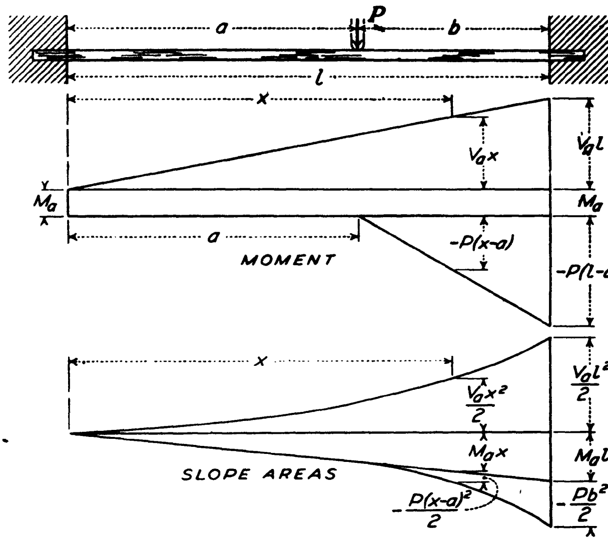


FIG. 171. General moment and separate slope diagrams.

107-2. Instead of Eq. (107.1), use the condition that the beam will have zero deflection at the left end measured from the tangent at the right end, and combine with Eq. (107.2) to solve for M_a.

107-3. A beam is fixed at both ends and loaded at the middle. By using the moment diagram directly and without substituting in the formulas above, find the moments and shears at the ends. Find also the maximum deflection.
Ans. $y_{max} = -Pl^3/192EI$.

107-4. Find the moment at each end, the maximum positive moment, and the location of points of contraflexure for a beam which is fixed at the ends and loaded at six-tenths the length from the left end of the span.
Ans. $M_a = -0.096Pl$; $M_b = -0.144Pl$; M at load $= 0.1152Pl$; contraflexure at $3l/11$ from left end, and $2l/9$ from right end.

107-5. Calculate the shear at the left end for Prob. 107-4. Write the general moment equation and calculate the deflection under the load by means of the deflection from the tangent at the left end.
Ans. $y = -0.004608Pl^3/EI$.

107-6. Find the point of maximum deflection for the beam of Prob. 107-4 and calculate the maximum deflection.
Ans. $y_{max} = -6.336Pl^3/1{,}331EI = -0.00476Pl^3/EI$.

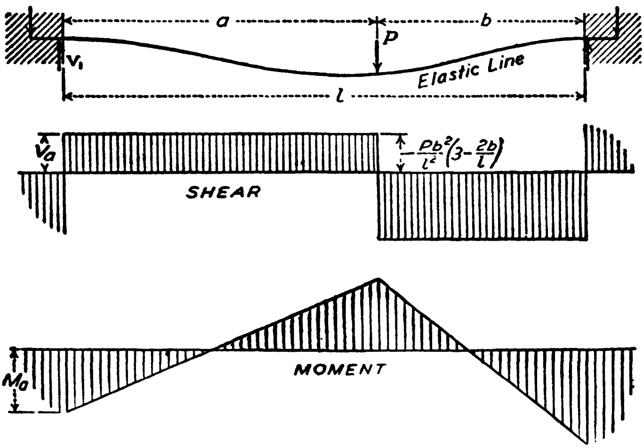


FIG. 172. Shear and resultant moment diagrams for concentrated load.

107-7. A beam 7 ft long is fixed horizontally at both ends. An 8,400-lb load is applied at 2 ft from the right end and an equal load at 2 ft from the left end. Working from the diagrams, find the maximum moment, neglecting the weight of the beam, and select a standard steel I beam to carry the loads. *Ans.* 6-in. 12.5-lb I beam.

107-8. Find the maximum deflection of the beam in Prob. 107-7.

107-9. How does the maximum deflection of a beam fixed at both ends and loaded at the middle compare with the deflection of a similar beam which is fixed at both ends and loaded uniformly (*a*) when the loads are equal, (*b*) when the maximum stresses are equal?

Example 1

A 5-in. 5.25-lb aluminum I beam 140 in. long is supported at 20 in. from the left end and at 30 in. from the right end. It carries 1,800 lb on the left end, 2,400 lb on the right end, and 6,000 lb at 50 in. from the left end. Neglect the weight of the beam and find the slope over the left support.

Only the moment diagram between supports need be drawn similar to that shown in Fig. 173. The left reaction is 5,400 lb and the right 4,800 lb. The moment $M_a = -1{,}800 \times 20 = -36{,}000$ in.-lb. From the shear diagram, $V_{ab} = +3{,}600$ lb. To find the deflection of the right support from the tangent over the left support, take moments about the right end of the moment diagram (Fig. 173,II).

Altitude	*Base*	*Area*	*Arm*	*Moment of moment diagram*
324,000	90	14,580,000	30	437,400,000
−36,000	90	−3,240,000	45	−145,800,000
−360,000	60	−10,800,000	20	−216,000,000

$$EIy = 10{,}600{,}000 \times 15.22y = +\ 75{,}600{,}000$$

At the right support $y = +0.4686$ in.; $\theta_L = -\dfrac{0.4686}{90} = -0.005206$ rad.

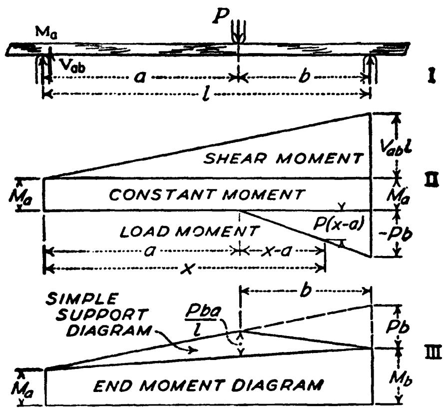


FIG. 173. Separate and combined diagrams for concentrated load.

Problem

107-10. In the example above, remove the 6,000-lb concentrated load and add 100 lb per ft over the 90 in. between the supports. Solve for the slope of the beam at the left support.

108. Theorem of Three Moments. The methods of the preceding articles may be applied to any number of spans and to any distribution of loads. When, as usually happens, it is desired to find the moments, reactions, and shears without calculating the deflections, the *theorem of three moments* is a valuable labor-saving device.

The theorem of three moments is an *algebraic equation* which expresses the relation of the moments at three successive supports of a continuous beam in terms of the lengths of the intervening spans and the loads which they carry. In Fig. 174, the moments over the supports are M_a, M_b, and M_c. The length of the span between support A and support B is l_1, and the length between support B and support C is l_2. The figure represents a uniformly distributed load of w_1 over the first span and w_2 over the second span.

The subscripts a, b, and c represent order from left to right and may be applied to *any three supports in succession.*

The shear adjacent to B on the right side toward C is V_{bc}. The shear on the side of B toward A is V_{ba}. In all cases shear is from the left side of the section to the right side.

When there are four supports, two equations are written. For the first equation, supports 1, 2, and 3 are represented by A, B, and C. For the second equation, 2, 3, and 4 follow in the same order.

109. Three Moments, Load Uniformly Distributed. When the second span of Fig. 174 is considered, the deflection of the beam at C may be found from the tangent at B. The trapezoidal moment diagram has been divided into two triangles.

$$EIy_c = \frac{M_bl_2}{2} \times \frac{2l_2}{3} + \frac{M_cl_2}{2} \times \frac{l_2}{3} + \frac{w_2l_2^3}{12} \times \frac{l_2}{2} \tag{109.1}$$

$y_c = \theta_2 l_2$, and since y is negative, θ is negative.

$$6EI\theta_2 = 2M_bl_2 + M_cl_2 + \frac{w_2l_2^3}{4} \tag{109.2}$$

which may be called the *two-moment equation.* A similar equation

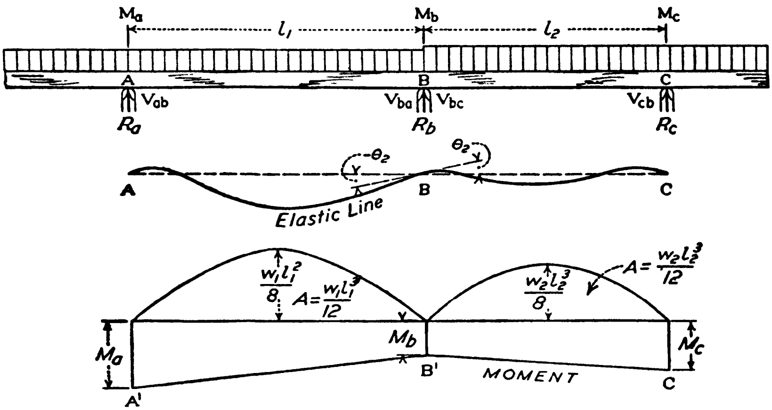


FIG. 174. End-moment and simple-support diagrams for two spans.

may be written for the first span, measuring the deflection of A from the tangent at B. Reducing to θ, this becomes

$$6EI\theta_2 = 2M_bl_1 + M_al_1 + \frac{w_1l_1^3}{4} \tag{109.3}$$

In the above equation, θ is positive because y was positive. When the last two equations are added, the terms on the left side cancel.

$$M_al_1 + 2M_b(l_1 + l_2) + M_cl_2 = -\frac{w_1l_1^3}{4} - \frac{w_2l_2^3}{4} \tag{109.4}$$

This is the theorem of three moments for uniformly distributed loads. It is understood, of course, that the three supports are at the same level. For equally loaded spans of equal length, $w_1 = w_2$, $l_1 = l_2$, and

$$M_a + 4M_b + M_c = -\frac{wl^2}{2} \tag{109.5}$$

Example 1

A beam of uniform weight w per unit length rests on four supports to form three equal spans, each of length l, and overhangs the left support $0.4l$ and the right support $0.2l$. Find the moment over each support.

From the overhanging ends,

$$M_1 = -0.4wl \times 0.2l = -0.08wl^2$$
$$M_4 = -0.2wl \times 0.1l = -0.02wl^2$$

Writing Eq. (109.5) for the first three supports,

$$-0.08wl^2 + 4M_2 + M_3 = -0.50wl^2$$
$$4M_2 + M_3 = -0.42wl^2$$

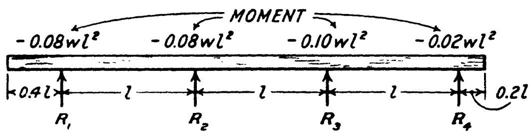


FIG. 175. Three equal spans.

For the second, third, and fourth supports,

$$M_2 + 4M_3 - 0.02wl^2 = -0.50wl^2$$
$$M_2 + 4M_3 = -0.48wl^2$$

Solving simultaneously, $M_2 = -0.08wl^2$

$$M_3 = -0.10wl^2$$

Example 2

A uniformly loaded beam 24 ft long is supported 6 ft from the left end, 16 ft from the left end, and at the right end. Find the moment over each support.

$$-18w \times 10 + 36M_2 + 0 = -\frac{w}{4}(1{,}000 + 512)$$
$$M_2 = -5.5w$$

Problems

109-1. A beam 20 ft long is supported 2 ft from the left end, 10 ft from the left end, and at the right end. It carries 60 lb per ft over the entire length and 30 lb concentrated at the left end. Find the moment over each support. *Ans.* −180 ft-lb; −590 ft-lb; 0.

109-2. A beam 16 ft long is supported at the ends and 6 ft from the right end. It carries 24 lb per ft over the 10-ft span and 40 lb per ft over the 6-ft span. Find the moment over the intermediate support. *Ans.* $M_b = -255$ ft-lb.

109-3. A beam 26 ft long is supported at 4 ft from the left end, 14 ft from the left end, and 2 ft from the right end. It carries 20 lb per ft over the left 14 ft and 36 lb per ft over the remainder. Find the moments over the supports. *Ans.* −160 ft-lb; −296 ft-lb; −72 ft-lb.

109-4. Solve Prob. 109-3 for M_b if there is no load over the left 10-ft span and the load over the remainder is not changed.

109-5. The beam of Fig. 174 overhangs the left support 5 in. and the right support 4 in. The left span is 12 in. long and the right span is 10 in. long. The load is 16 lb per in. over 17 in. from the left end to the intermediate support and 18 lb per in. over the remaining 14 in. There is a concentrated load of 31 lb 3 in. from the left end and another of 29 lb 2 in. from the right end. Find the moment over each support.

(These data were used in calculating the elastic line of Fig. 174.)

Ans. $M_a = -262$ in-lb.; $M_b = -142$ in-lb.; $M_c = -202$ in-lb.

109-6. A uniformly loaded beam rests on three supports to form two equal spans with equal overhang on each end. What must be the ratio of the overhang to the length of a span if the moment is the same at each support? What is this moment? *Ans.* $a/l = 0.408$; $M = -wl^2/12$.

109-7. Solve Prob. 109-6 for three spans.

110. Reactions by Moments. After the moments over the supports have been calculated by the theorem, the reaction of any support may be determined by equating the known moment at an adjacent support to the moment about that support of the unknown reaction and of all other known forces which act on the portion of the beam which is taken as the free body.

Example

A beam 24 ft long is supported at 4 ft and 16 ft from the left end and at 2 ft from the right end. It carries a concentrated load of 120 lb on the left end, and a concentrated load of 240 lb on the right end. There is a uniform load of 40 lb per ft distributed over the portion of the beam between the left and middle support, and 80 lb per ft distributed between the right and middle support. Find the reactions.

$$M_a = -480 \text{ ft-lb} \qquad M_c = -480 \text{ ft-lb}$$

$$-480(12) + 2(12 + 6)M_b - 480(6) = -\tfrac{1}{4}(40 \times 1{,}728 + 80 \times 216)$$

$$M_b = -360 \text{ ft-lb}$$

Take moments about the middle support of all the forces to the left.

$$12R_a - 120(16) - 40(12)(6) = M_b = -360$$

$$R_a = 370 \text{ lb}$$

Take moments about the middle support of all the forces to the right.

$$6R_c - 240(8) - 80(6)(3) = M_b = -360$$

$$R_c = 500 \text{ lb}$$

Take moments about the right support of everything to the left.

$$370(18) + 6R_b - 120(22) - 40(12)(12) - 80(6)(3) = M_c = -480$$

$$R_b = 450 \text{ lb}$$

Problems

110-1. Find the reactions for Prob. 109-1. *Ans.* $R_c = 241$ lb.

110-2. Find the reactions and draw the shear diagram for the beam of Prob. 109-2. Compute all maximum moments. If a square timber beam is used to support these loads, find the size. *Ans.* $R_a = 94.5$ lb; $d = 2.49$ in.

110-3. In Prob. 109-3, find the reactions, draw the shear diagram, find the maximum moments and compute the size of a square timber beam to support the loads. *Ans.* $R_a = 166.4$ lb.

110-4. Find the reactions for Prob. 109-5. *Ans.* 217 lb; 170 lb; and 197 lb.

110-5. Find the reactions for Prob. 109-6. *Ans.* 150 lb; 104 lb; and 156 lb.

111. Reactions by Shear. When there are more than three supports, the calculations of reactions by the method of Art. 110 becomes laborious. A more general method, applicable to any number of spans, is based on the difference between the shear on opposite sides of the support.

Infinitely close to the right of support A of Fig. 174, the shear as given by Eq. (103.3) is

$$V_{ab} = \frac{M_b - M_a}{l} + \frac{wl}{2} \tag{111.1}$$

A similar expression applies at the right of the second support. When the shear is known at the left of any span from Eq. (111.1), the shear at the right end of the span is found by subtracting the intervening load. The reaction of any support is found by subtracting the shear at the left from the shear at the right. Usually, the shear is negative at the left side and positive at the right, which makes the algebraic difference the sum of the two shears.

Example

To find the shear at the right side of support A of Fig. 174 from the data of Prob. 109-5,

$$V_{ab} = \frac{-142 + 262}{12} + 96 = 106 \text{ lb}$$

At the left side of support A by definition of shear,

$$V_{ao} = -80 - 31 = -111$$
$$R_a = 106 - (-111) = 217 \text{ lb}$$

Problems

111-1. A beam is supported at the ends and at three intermediate points such that there are four equal spans and no overhang. The beam carries a uniform load over its entire length. Find the moments at the supports and draw the shear diagram.

111-2. Solve Prob. 111-1 for an overhang of $0.4l$ at the left end and a uniform load of w per unit length over the four spans and $1.5w$ per unit length on the overhang. *Ans.* $-0.12wl^2$; $-0.075wl^2$; $-0.08wl^2$; $-0.105wl^2$; 0.

111-3. A beam 38 ft long is supported at 3 ft and 15 ft from the left end, and at 3 ft and 15 ft from the right end. It carries 40 lb per ft over the left 15 ft and also over the right 15 ft of its length, and 60 lb per ft over the middle 8 ft of the beam. Find the moments over the supports, draw the shear diagram, and locate all dangerous sections.

111-4. The beam of Fig. 176 carries a load w per unit length over the first, third, and fourth span, and a load of $2w$ over the second span. Find the moment over each support. Find the reactions.

Ans. Moments $= (-wl^2/32) \times (0, 5, 4, 3, 0)$
Reactions $= (wl/32) \times (11, 54, 48, 34, 13)$

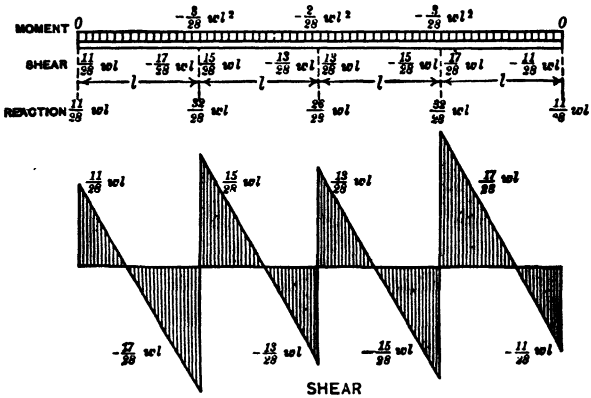


FIG. 176. Beam of four spans.

111-5. A beam of four equal spans carries a load of w per unit length over the second span and no load on the others. If the supports are hinged to the beam so that they may exert downward as well as upward reactions, find the moment at each. Find the reactions.

Ans. Moments = 0; $-11wl^2/224$; $-12wl^2/224$; $+3wl^2/224$; 0.

111-6. Find the moments and reactions for a uniformly loaded beam of five equal spans.

112. Three Moments, Loads Concentrated. The derivation of an equation for concentrated loads follows the pattern of Art. 108. In Fig. 177 the end-moment trapezoids are omitted (see Fig. 174). After the deflection at C from a tangent at B is found and divided by the length, an equation for the slope at B is obtained:

$$6EI\theta = 2M_bl_2 + M_cl_2 + \frac{Qc(l_2^2 - c^2)}{l_2} \tag{112.1}$$

As before, this may be called the *equation of two moments.* It is sometimes useful, as will be seen in the next article.

Writing from B in the opposite direction gives a similar equation for the first span. Addition of these two equations gives the theorem of three moments for a single concentrated load on each span.

$$M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2 = -\frac{Pa(l_1^2 - a^2)}{l_1} - \frac{Qc(l_2^2 - c^2)}{l_2} \qquad (112.2)$$

The beam reactions may be found again as in Art. 110, or the shear as in Art. 111. The following example will illustrate.

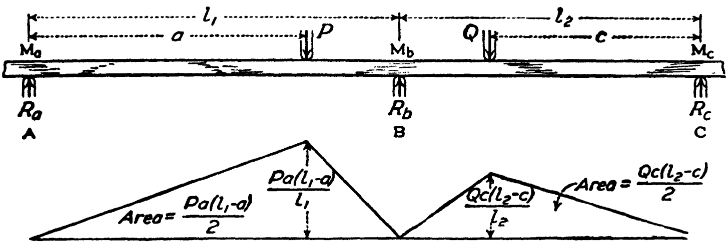


FIG. 177. Simple-support diagrams for concentrated loads.

Example

A beam 23 ft long is supported 2 ft from the left end, 12 ft from the left end, and 3 ft from the right end. It carries 90 lb on the left end, 96 lb on the right end, 120 lb 6 ft from the left end, and 240 lb 5 ft from the right end. Find the moments and reactions. Draw the shear diagram and find the maximum moment on each span.

$$-180 \times 10 + 36M_2 - 288 \times 8 = -\frac{120 \times 4 \times 84}{10} - \frac{240 \times 2 \times 60}{8}$$

$$\begin{aligned} 36M_2 &= 1{,}800 - 4{,}032 \\ &\ \underline{2{,}304} - \underline{3{,}600} \\ 36M_2 &= 4{,}104 - 7{,}632 = -3{,}528 \\ M_2 &= -\ \ 98 \end{aligned}$$

$$V_{ab} = \frac{-98 + 180}{10} + \frac{6}{10} \times 120 = 8.2 + 72 = 80.2 \text{ lb}$$

$V_{ba} = 80.2 - 120 = -39.8$ $\quad V_{bc} = 36.25$ $\quad V_{cb} = -203.25$

$R_a = 80.2 - (-90) = 170.2$ $\quad R_b = 76.05$ $\quad R_c = 299.75$

$M_{max} = 140.8$ ft-lb in first span $\quad M_{max} = 119.5$ ft-lb in second span

Problems

112-1. A beam 35 ft long is supported 3 ft from the left end, 15 ft from the left end, 23 ft from the left end, and 2 ft from the right end. It carries 80 lb on the left end, 50 lb on the right end, 120 lb 7 ft from the left support, 160 lb in the second span 5 ft from the second support, and 200 lb 6 ft from the fourth support. Solve for the moments and reactions.

Ans. $M_3 = -266.45$ ft-lb; $R_1 = 139.71$; $R_2 = 102.43$.

112-2. A beam 30 ft long is supported 4 ft from the left end, 14 ft from the left end, and 2 ft from the right end. It carries 120 lb per ft, 600 lb 3 ft to the right of the left support, and 840 lb 9 ft to the right of the intermediate support. Find the moment over each support and find the reactions.

Ans. $M_2 = -3{,}480$ ft-lb; $R_1 = 1{,}248$ lb.

113. Continuous Beams, Ends Fixed.

When an end of a continuous beam is fixed, as in Fig. 178, the equation of two moments may

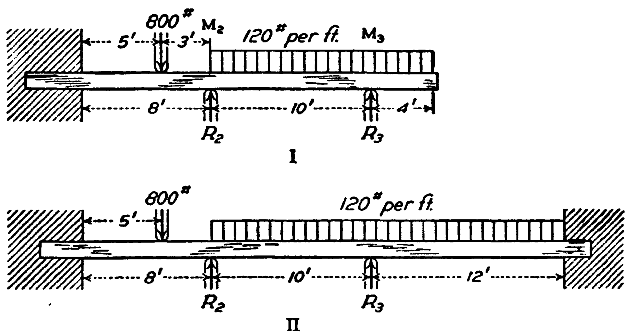


Fig. 178. Continuous beams with one or both ends fixed.

be written for the span adjacent to the fixed end, and the equation of three moments may be written for this span and the one next to it.

Example 1

Find the moments at R_2, R_3, and the wall for the beam of Fig. 178,I. For the 8-ft span, the equation of two moments, starting from the left end, at which the slope is zero, gives

$$8 \times 2M_1 + 8M_2 + \frac{800 \times 3 \times 55}{8} = 0 = EI\theta \qquad (113.1)$$

$$16M_1 + 8M_2 = -16{,}500$$

For the first two spans, the theorem of three moments gives

$$8M_1 + 36M_2 - 960 \times 10 = -\frac{800 \times 5 \times 39}{8} - \frac{120 \times 1{,}000}{4} \qquad (113.2)$$

$$8M_1 + 36M_2 = -39{,}900$$

$$M_1 = -536.72 \qquad M_2 = -989.06 \text{ ft-lb}$$

The student should find the reactions, draw the shear diagram, locate the dangerous sections, and find the maximum moments which occur between the supports. A beam can then be selected to carry these loads.

This problem may also be worked by reflecting an image of the beam from the wall to the left. The beam then has five supports and by symmetry is horizontal over the middle one. By writing the three-moment equation twice, the moments over the supports are obtained.

A third solution is possible by removing the wall and adding two vertical sup-

ports very close together. Calling these R_0 and R_1 and the length between them zero, we have, by the three-moment equation, remembering that $M_0 = 0$,

$$0 \times 0 + 2M_1(0 + 8) + M_2 \times 8 = -\frac{800 \times 3 \times 55}{8} \tag{113.3}$$

This checks Eq. (113.1). The second three-moment equation is given by Eq. (113.2).

Example 2

The beam of Fig. 178,I is extended to form a third span (Fig. 178,II) which is 12 ft long, is fixed at the right end, and is uniformly loaded with 120 lb per ft. Find the moment at the fixed ends and at the supports.

Equation (113.1) is still valid.

Equation (113.2) is now changed to read

$$8M_1 + 36M_2 + 10M_3 = -\frac{800 \times 5 \times 39}{8} - \frac{120 \times 1{,}000}{4} \tag{113.4}$$

For the second and third spans,

$$10M_2 + 44M_3 + 12M_4 = -81{,}840 \tag{113.5}$$

For the third span alone, using Eq. (109.2),

$$12M_3 + 24M_4 + \frac{120 \times 1{,}728}{4} = 0 \tag{113.6}$$

$$M_1 = -579.5 \qquad M_2 = -903.6 \qquad M_3 = -1{,}233.8 \qquad M_4 = -1{,}543.1$$

The method of reflection, mentioned under Example 1, cannot be used here.

The third method, replacing each wall by two supports and writing the three-moment equation four times, gives a satisfactory solution.

Problems

113-1. A beam fixed at both ends has an intermediate support. The left span is 8 ft long and carries 240 lb per ft. The right span is 12 ft long and carries 90 lb per ft. Find the moments at the ends of the spans.
Ans. $M_1 = -1{,}340$ ft-lb; $M_2 = -1{,}160$ ft-lb; $M_3 = -1{,}040$ ft-lb.

113-2. In Prob. 113-1, find the shear at the fixed ends and the reaction of the support. *Ans.* $V_{12} = 982.5$ lb; $R = 1{,}387.5$ lb.

113-3. Find the maximum positive moments for Prob. 113-1.
Ans. 671.08 ft-lb at $4\frac{3}{32}$ ft from left end;?

113-4. A beam fixed at both ends has an intermediate support, which makes the left span 10 ft and the right span 8 ft. There is a load of 400 lb 3 ft to the left of the support and a load of 640 lb 3 ft to the right of the support. Find the moments at the five dangerous sections.
Ans. −216; 313.2; −660; 601.9; −495 ft-lb.

113-5. A beam fixed at both ends has two intermediate supports. The left span is 8 ft long and carries 120 lb per ft. The second span is 12 ft and carries 180 lb per ft. The third span is 10 ft long and carries 100 lb per ft. Find the moment at each of the seven dangerous sections.

114. Three Moments, Spans Partially Loaded. When the uniform load covers only a part of the length between supports, an equation may be derived by starting with the concentrated load equation, (112.2). Let the load $P = w\,dx$ and the distance a be x. The same can be done for Q and c, so that the right side of Eq. (112.2) will be

$$-\int_a^b \frac{w_1\,dx(x)(l_1^2 - x^2)}{l_1} - \int_c^d \frac{w_2\,dx(x)(l_2^2 - x^2)}{l_2} \qquad (114.1)$$

where a is the distance from the *first* support to the beginning of the uniform load in the first span and b is the distance from the first support to the right end of the load. In the second span c is the distance from the *third* support to the far end of the uniform load, and d is measured from the third support to the left end of the uniform load in the second span.

$$M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2 = -\frac{w_1(b^2 - a^2)}{4l_1}[2l_1^2 - (b^2 + a^2)] - \frac{w_2(d^2 - c^2)}{4l_2}[2l_2^2 - (d^2 + c^2)] \qquad (114.2)$$

Example

The beam of Fig. 179 is fixed at the left end, is supported at three points, and overhangs the right support 4 ft. The left span is 8 ft long and uniformly loaded

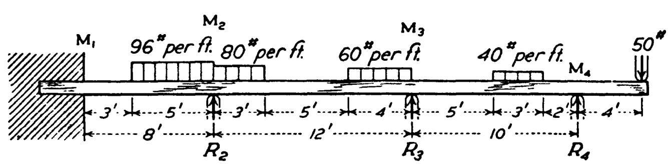


FIG. 179. Continuous beam with partially loaded spans.

with 96 lb per ft for 5 ft adjacent to the support. The second span is 12 ft long. It carries 80 lb per ft over the left 3 ft and 60 lb per ft over the right 4 ft. The third span is 10 ft long. It carries 40 lb per ft over 3 ft which begins 5 ft from the left end. A load of 50 lb is placed at the right end of the overhang. Find the moment at each support and at the fixed end.

When the two-moment equation is applied to the first span, $l_1 = 8$, $c = 0$, $d = 5$.

$$16M_1 + 8M_2 + \frac{96 \times 25(128 - 25)}{32} = 0$$

$$16M_1 + 8M_2 + 7{,}725 \text{ ft-lb} = 0$$

When the three-moment equation is applied to the first two spans, $l_1 = 8$; $a = 3$; $b = 8$; $l_2 = 12$, $c = 9$, $d = 12$, for the 80-lb load; $l_2 = 12$, $c = 0$, $d = 4$, for the 60-lb load

$$8M_1 + 40M_2 + 12M_3 + \frac{96 \times 25(8 - 2.5)^2}{8} + \frac{80 \times 9(12 - 1.5)^2}{12} + \frac{60 \times 16(288 - 16)}{48} = 0$$

$$8M_1 + 40M_2 + 12M_3 + 21{,}130 = 0$$

$$12M_2 + 44M_3 - 2{,}000 + \frac{80 \times 9(288 - 9)}{48} + \frac{60 \times 16 \times 100}{12} + \frac{40 \times 25 \times 175}{40} - \frac{40 \times 4 \times 196}{40} = 0$$

$$12M_2 + 44M_3 + 13{,}776 = 0$$

$$M_1 = -276.4 \qquad M_2 = -412.8$$
$$M_3 = -200.5$$

Problem

114-1. Find the shear at the left end of the beam of above example and the reaction at each support.

115. Deflection from Tangent. Figure 180 shows the elastic line for any span between supports. A tangent is drawn through any point C on this curve. The

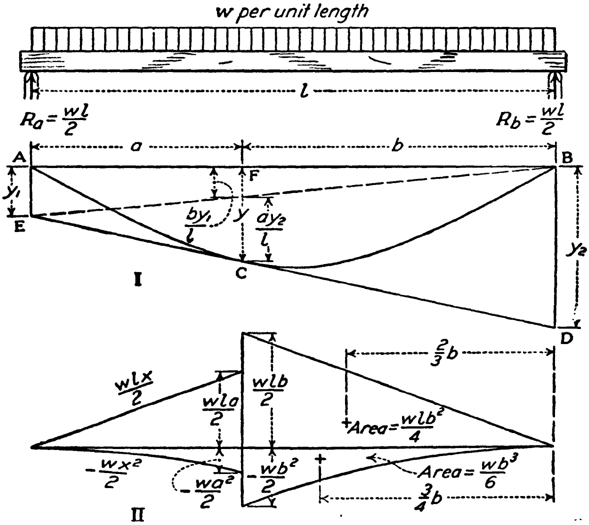


FIG. 180. Deflection of supports from tangent.

vertical distance EA upward from this tangent to the left end of the span is y_1 and the corresponding distance DB at the right end is y_2. The distance CF upward to the horizontal line which connects A and B is given by

$$CF = y_1 + \frac{a}{l}(y_2 - y_1) = y_1 + \frac{a(y_2 - y_1)}{a + b} \tag{115.1}$$

$$CF = \frac{ay_1 + by_1 + ay_2 - ay_1}{a + b} = \frac{by_1 + ay_2}{a + b} = -y \tag{115.2}$$

in which y is the deflection of C from the line of the supports, a is the horizontal distance from the left support to C, and b is the horizontal distance from the right support to C. [Equation (115.2) may be written geometrically from Fig. 180 by means of the broken line EB.]

Figure 180,II is the moment diagram for a uniformly loaded, simply supported beam. The moment is calculated from each end to any point C. Each portion of the beam is a cantilever, which is bent upward by the reaction and bent downward by the uniformly distributed load. By using the equations for the deflection at the ends,

$$EIy_1 = \frac{wla^3}{6} - \frac{wa^4}{8} \qquad EIy_2 = \frac{wlb^3}{6} - \frac{wb^4}{8} \tag{115.3}$$

in which y_1 and y_2 are measured upward. By substitution in Eq. (115.2), with $a + b = l$, Eq. (115.3) becomes

$$EIy = -\frac{wab}{6}(a^2 + b^2) + \frac{wab}{8l}(a^3 + b^3) \tag{115.4}$$

This is the form of the elastic-line expression which is most convenient to derive by the methods of elastic energy.

Example

Find the deflection of a uniformly loaded, simply supported beam at one-third the length from the left end. Calculate y_1 and y_2 by the equations for the deflection at the end of a cantilever and substitute in Eq. (115.2).

$$\begin{aligned}
EIy_1 &= \frac{1}{3} \times \frac{wl}{2}\left(\frac{l}{3}\right)^3 - \frac{w}{8} \times \frac{l^4}{3} = \frac{3wl^4}{648} \\
EIy_2 &= \frac{1}{3} \times \frac{wl}{2} \times \left(\frac{2l}{3}\right)^3 - \frac{w}{8} \times \left(\frac{2l}{3}\right)^4 = \frac{16wl^4}{648} \\
-y &= \frac{wl^4}{648EI} \times \frac{3 \times 2l/3 + 16 \times l/3}{l} \\
y &= -\frac{11wl^4}{972EI}
\end{aligned}$$

Problems

115-1. Solve the example for the deflection at one-fourth the length from each end. *Ans.* $y = -19wl^4/2{,}048EI$.

116. Miscellaneous Problems

116-1. A beam is 20 ft long and rests on three supports, at 4, 12, and 18 ft from the left end. There is 60 lb per ft over all the beam except the right 2 ft. There is a 200-lb concentrated load at 10 ft from the left end, and a 285-lb load at 6 ft from the right end. Find the moments over the supports. Find the reactions. Find the maximum positive moment in the first span.
Ans. $M_1 = -480$; $M_2 = -500$; $M_3 = -180$; $R_1 = 527.5$; $M = +448.8$.

116-2. A beam 22 ft long is supported at 4 ft and at 10 ft from the left end, and at 3 ft from the right end. It carries 120 lb per ft from the left end to the

middle support, and 80 lb per ft from the middle support to the right end. Draw the shear diagram for the beam.

116-3. A weightless beam 12 ft long is supported at 4 ft from the left end and at 3 ft from the right end. It carries 150 lb on the left end and 200 lb on the right end. If the beam is a 6- by 2-in. wood beam, find the deflection of a point midway between the supports, using area-moment methods. Find the maximum bending stress in the beam.

116-4. A weightless beam 10 ft long is supported at the left end and fixed at the right end. EI for the beam is 300,000 lb-in.^{2} At a point 4 ft from the left end, a clamp is attached and a counterclockwise moment of 500 in.-lb applied. Find the reaction at the left end and the slope of the beam at the left end.

116-5. A weightless beam 40 in. long is fixed at both ends and carries 72 lb per ft over the middle 20 in. with no load on the 10 in. adjacent to each end. Find the three maximum moments.

116-6. A weightless rectangular 4- by 3-in. wood beam is fixed in a horizontal position between walls 5 ft apart. When a 1,000-lb concentrated load is placed 2 ft from the left end, the left end of the beam slips and assumes a slope equal to 0.012 rad. Find the end moments and shears.

116-7. If the length of the span of Fig. 181 is l and each end overhangs $0.1l$, find the slope at each end when a load of $0.4P$ is placed on the left end,

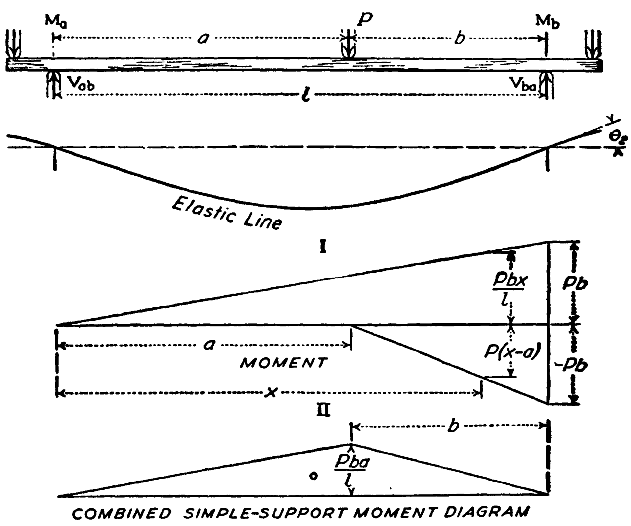


FIG. 181. Elastic line and moment diagram.

a load of $0.16P$ is placed on the right end, and a load P is placed $0.6l$ from the left support. Check by means of the area of the end-moment diagram (not drawn) and the combined simple-support diagram.

Ans. $EI\theta_2 = +0.052Pl^2$.

116-8. The uniformly loaded beam of Fig. 182 is fixed at the right end and supported 60 in. from the fixed end. The beam overhangs the left support 20 in. Find the reaction, the maximum moments, and the slope over the support by means of the separate diagrams shown in the figure.

Ans. $R = 47.5w$; $\theta_1 = -1{,}500w/EI$.

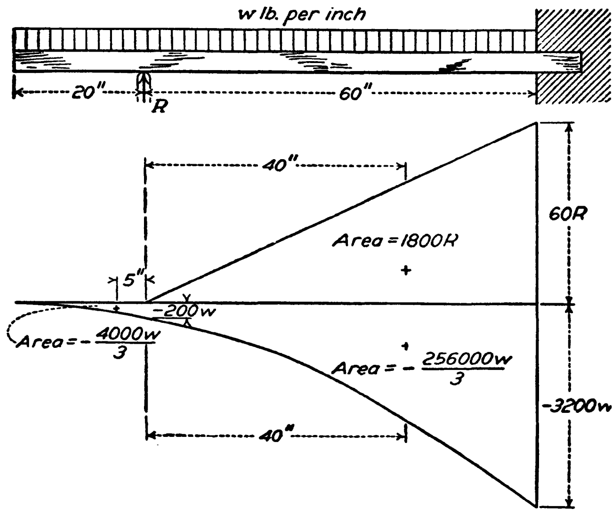


FIG. 182. Separate moment diagrams for overhanging beam.

116-9. Work Prob. 116-8 by area moments, using the general moment diagram instead of Fig. 182.

116-10. Solve for V_{ab} in Fig. 183 by the general moment equation as shown by the moment diagram, using area moments. Calculate the moment at the fixed end and the slope over the support.

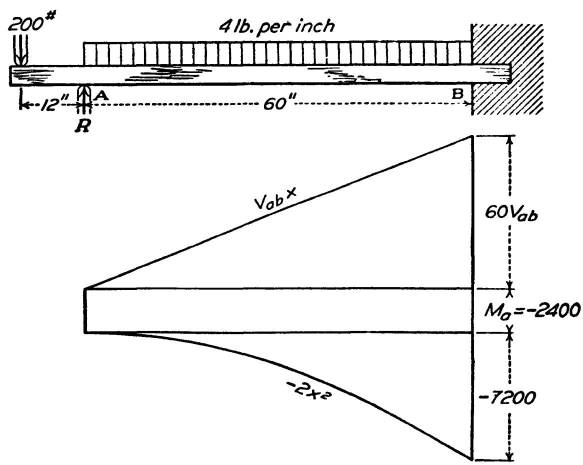


FIG. 183. General moment diagrams.

116-11. A beam 10 in. long between supports overhangs the left support 4 in. and the right support 2 in. There is a uniformly distributed load of 12 lb per in. over the entire length, a load of 30 lb on the right end, and a load of 60 lb 7 in. from the left support. Find the slope at the left end of the span. *Ans.* $\theta_1 = -313/EI$.

116-12. Using the answer of Prob. 116-11, find the slope at the right end of the span by means of the area of the moment diagram.

116-13. Find the deflection of the beam of Prob. 116-11 at the middle of the span. Find the deflection at the free ends by means of the slope at the supports and the cantilever formulas.
Ans. $-1{,}427.5/EI$; $+868/EI$; $+730/EI$.

116-14. Find the points of inflection for the beam of Prob. 116-11.
Ans. 1.35 in., and 9.12 in. from the left support.

116-15. A beam fixed at the left end is supported at a distance l from the left end and overhangs the support $0.4l$. It carries a uniformly distributed load of w per unit length. Find the moment at the fixed end.
Ans. $M_1 = -0.21wl^2$.

116-16. A beam 5 ft long is fixed at the right end and supported at the left end. It carries a load which increases in intensity from zero at the support to 20 lb per in. at the wall. Find the moments at the dangerous sections.
Ans. 2,146.4; $-4{,}800$. ft-lb.

116-17. A beam weighing 12 lb per ft is fixed at the left end and supported 10 ft from the fixed section. It overhangs the support 6 ft and carries 36 lb 1 ft from the free end and 60 lb 6 ft from the fixed section. Find the moment at the fixed end. Find the shear at the fixed end.
Ans. $M_1 = -528$ ft-lb.

116-18. A 10- ft span is fixed at the left end and supported at the right end. The load increases uniformly from 20 lb per ft at the left end to 80 lb per ft at the support. Find the moment and shear at the fixed end and the reaction of the support. *Ans.* $M = -600$ ft-lb; $V = 260$ lb; $R = 240$ lb.

116-19. In Prob. 116-18, find the maximum positive moment.
Ans. $M = 393$ ft-lb.

116-20. A span 10 ft long is fixed at the left end and supported at the right end. At 4 ft from the left end the load is 20 lb per ft and increases uniformly from that point to 68 lb per ft at each end. Find the moment at the fixed end, the reaction at the support, and the maximum positive moment. Solve for the moment by a uniform load of 20 lb per ft and two uniformly increasing loads.
Ans. $M = -471.12$ ft-lb; $R = 180.888$ lb; $M_{max} = 274.6$ ft-lb.

116-21. A horizontal beam 21 ft long is supported at 2, 10, and 18 ft from the left end. It carries a concentrated load of 300 lb on the left end, and uniform loads as follows: 80 lb per ft over the right 3 ft of the beam, and 160 lb per ft over the span between the left and middle supports. Draw the shear diagram and find all maximum moments. *Ans.* $R_2 = 620$ lb.

116-22. A beam, originally horizontal, is 90 in. long. It is fixed at the right end and supported on a roller at 60 in. from the fixed end. A 100-lb load is placed on the free end. When a load P is applied at 40 in. from the wall, the 100-lb load is elevated until the beam at the free end becomes hori-

zontal. If $EI = 10{,}000{,}000$ lb-in.^{2} find the load P and the reaction at the support, using area-moment methods. *Ans.* $R = 525$ lb; $P = 675$ lb.

116-23. In Prob. 116-22 find the place, between the reaction and the wall, where the beam is horizontal. *Ans.* 54 in. from the end.

116-24. In Prob. 116-22 draw the shear and moment diagrams and find the maximum bending moments.

116-25. An 8- by 3-in. rectangular beam, originally horizontal, is 15 ft long. Both ends are firmly clamped and the middle of the beam rests on a support. When two 400-lb loads are applied, each load being 5 ft from the wall, the middle support settles ½ in. If $E = 1{,}200{,}000$ psi, find the reaction at the support. *Ans.* 237 lb.

116-26. Draw the shear diagram for the beam of Prob. 116-25. Find the maximum moments and the maximum bending stress in the beam. Locate the points of contraflexure.

CHAPTER 12

BENDING COMBINED WITH DIRECT STRESS

117. Transverse and Longitudinal Loading. A beam may be subjected to an *axial* (*in the direction of its length*) tension or compression and transverse forces which produce bending moments. Since the bending moments cause axial stresses these stresses may be added to the direct tensile or compressive stresses which are caused by the axial loads.

In Fig. 184, a 4- by 4-inch vertical post is fixed at the bottom and carries a load of 4,000 pounds on the top. The direct compressive stress is 250 pounds per square inch in all parts of the post between the load and the support. (This assumes that the post has only a negligible deflection due to the 200-pound force.) A horizontal force, perpendicular to one face, is applied 2 feet above the plane at which the post is fixed. The transverse force causes a tensile stress of 450 pounds per square inch at the fixed plane on the right side and an equal compressive stress on the left side. The combined stress is 700 pounds per square inch compression on the left side and 200 pounds per square inch tension on the right side. Figure 184,IV shows the distribution of stress, with compression represented by a negative ordinate and tension by a positive ordinate. Figure 184,II shows the compression alone, which is caused by the direct load of 4,000 pounds. Figure 184,III shows the stress, which is caused by bending. This stress is 450 pounds compression at the left, 450 pounds tension at the right, and zero at the middle. These stresses are combined in Fig. 184,IV. The line *EF*, which is the zero line for the bending stress, is placed on the line *CD*, which represents the compressive stress in Fig. 184,II.

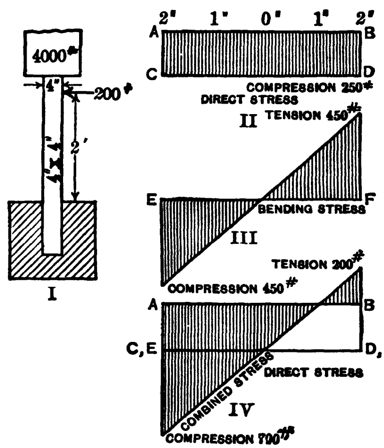


FIG. 184. Post with compression and bending.

The combined unit compressive stress on the left side is

$$-250 - 450 = -700 \text{ pounds per square inch}$$

On the right side the combined stress is $-250 + 450 = 200$ pounds per square inch tension. The unit stress is zero at $\frac{8}{9}$ inch from the right face of the post.

The unit stress is

$$\frac{P}{A} + \frac{Mv}{I} \qquad S = \frac{P}{A} \pm \frac{M}{Z} \text{ for outer fibers} \tag{117.1}$$

in which P is the total axial load and M is the bending moment from any source whatever. Since v is positive on one side of the neutral axis and negative on the other, the second term of the equation may be positive or negative, depending upon the position of the fiber in the section.

Problems

117-1. A timber post, 6 in. square, is fixed at the bottom with two parallel faces in the meridian (north and south vertical plane). The post is 4 ft high from the fixed section. It carries a load of 10,800 lb on the top and resists a horizontal force of 360 lb pushing west 6 in. from the top. Find the unit stress in the east and west outer fibers at the bottom.
Ans. 720 psi compression and 120 psi tension.

117-2. In Prob. 117-1, how far from the bottom is the stress zero in the extreme eastern fibers? *Ans.* 12 in.

117-3. In Prob. 117-1, how far from the east surface at the bottom is the stress zero?

117-4. A 6- by 8-in. rectangular post 5 ft high has its 8-in. edges in the meridian. A 9,600-lb load is placed on top and a horizontal 400-lb force pushes east at the middle of the 8-in. edge at the top. Find the maximum stresses at the bottom. *Ans.* 700 psi compression; 300 psi tension.

117-5. In Prob. 117-4, how far from the bottom on the west side is the stress zero in the extreme west fibers? *Ans.* 36 in.

117-6. In Prob. 117-4, how far from the west edge at the bottom is the stress zero? *Ans.* 1.8 in.

117-7. In Prob. 117-4, find the equation of the surface on which the stress is zero. HINT: The answers to Probs. 117-5 and 117-6 must satisfy the equation.

117-8. A 6- by 8-in. rectangular post 61 in. high has its 8-in. edges in the meridian. When a 3,660-lb load is placed on the middle of the top, the post settles unequally so that the north 6-in. edge is 11 in. farther north than the bottom 6-in. edge (which was directly underneath originally). Find the stresses in the outer fibers at the bottom. *Ans.* 75 ± 629 psi.

117-9. A rectangular pier is 6 ft wide, 3 ft thick, and 50 ft high above the ground. If the material weighs 120 lb per cu ft and is unable to resist tension, what maximum wind pressure uniformly distributed over one entire face will be safe? Use all units in feet. *Ans.* 7.2 lbs per sq ft.

117-10. Solve Prob. 117-9 if the material has a tensile strength of 10 psi.

118. Eccentric Loading. Figure 185 represents a rigid bar G supported by three equal rubber bands (or springs) which are symmetrically placed and suspended from a rigid horizontal support. Each of the bands is stretched the same amount and the bar hangs in a horizontal position. Figure 185,II shows the same bar with a load P at the middle. The rubber bands are equally stretched and the bar remains in a horizontal position. If the load P be moved to the right, as in Fig. 185,III, the middle band receives the same elongation as in the preceding case, while the left band is elongated less and the right band more. If the load is moved still farther to the right, a place is reached where the left end is elevated above the position which it occupied before the load was applied, so that no load whatever comes on the left band. If, instead of the rubber bands, helical springs were used, the spring on the side away from the load would come into compression when the load is shifted a considerable distance from the middle.

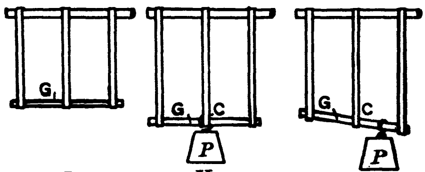


FIG. 185. Eccentric loading of rubber bands.

Figure 185,III shows that the effect of the eccentric load is a translation downward, of the same magnitude as that caused by the central load in II, together with a rotation about the bottom of the middle band as an axis.

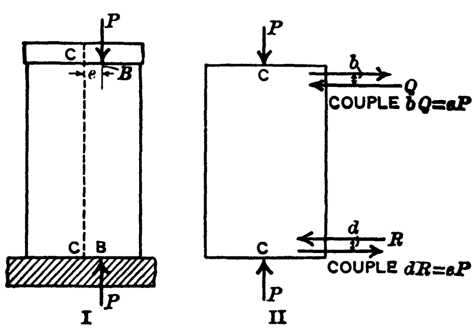


FIG. 186. Block with eccentric loading.

Figure 186,I shows a block which is subjected to a load P at a distance e from its axis. A force along any line may be replaced by an equal force along any parallel line and a couple the moment of which is equal to the product of the force by the distance between the lines.[1]

[1] See any textbook of mechanics.

In Fig. 186, the force P at the top, at a distance e from the axis, may be replaced by a force P along the axis and a clockwise couple of moment $e \times P$. The reaction at the bottom may likewise be regarded as equivalent to a reaction P along the axis and a counterclockwise couple of moment $e \times P$. These equal couples are shown in Fig. 186,II. An eccentric load may be regarded as equivalent to a load along the axis combined with a bending moment, which is the product of the load multiplied by the eccentricity.

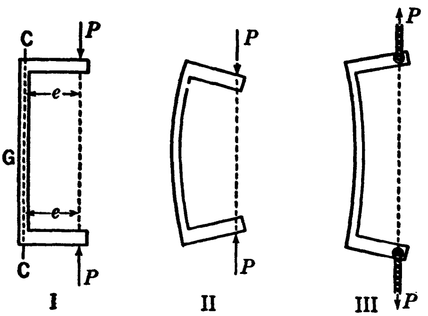


FIG. 187. Large eccentricity.

Figure 187 shows an example of large eccentricity. The portion above G may be treated as a free body. A vertical resolution shows that the direct force is P. The bending moment about G is Pe. When the deflection is caused by compression as in Fig. 187,II the eccentricity is increased. Tension reduces the eccentricity.

Since an eccentric load is equivalent to a central load and couple of moment $e \times P$, the unit stress at a distance v from the center of gravity of any section is

$$s = \frac{P}{A} \pm \frac{Mv}{I} \tag{118.1}$$

In these formulas, $M = e \times P$ is the moment of the load about the axis through the center of gravity of the section, v is the distance from the center of gravity of the section to any given fiber.

Problems

118-1. A short timber block, 6 in. square, is subjected to a load of 18,000 lb. The line of the resultant force is $\frac{3}{4}$ in. from the axis in a plane midway between the parallel faces. Find the maximum and minimum compressive stress. *Ans.* 875 psi; 125 psi.

118-2. Solve Prob. 118-1 if the force is 1 in. from the axis. *Ans.* 1,000 psi; 0.

118-3. Solve Prob. 118-1 for an eccentricity of 2 in. How far is the point of zero stress from the nearest face? *Ans.* 1.5 in.

118-4. A solid cylinder, 2 in. in diameter, is subjected to a pull of 15,708 lb along a line 0.25 in. from the axis. Find the maximum and minimum unit stress. *Ans.* 10,000 psi; 0.

118-5. In Prob. 118-4, how far from the surface is the tensile stress, 4,000 psi? Solve by the equation and by interpolation. *Ans.* 0.8 in.

118-6. Solve Probs. 118-4 and 118-5 for an eccentricity of 0.3 in.

118-7. Solve Prob. 118-4 if the cylinder is hollow with an inside diameter of 1 in.
Ans. 12,000 psi tension; 1,333 psi tension.

118-8. A short piece of 10-in. 15.3-lb standard channel carries a load of 10,000 lb. The resultant load lies in the plane of symmetry 1 in. from the back of the web. Find the maximum and minimum stress.
Ans. 5,237 psi compression at tip of flanges; 1,235 psi compression at back of web.

118-9. In a hook of circular section the distance from the center of gravity of the section to the line of the load is 3 in. The load is 1,600 lb and the diameter of the section is 2 in. Find the *approximate* value of the maximum tensile and compressive stress.
Ans. 6,621 psi tension; 5,602 psi compression.

118-10. In Fig. 187,III the clamp has a 2-in. width at G and a thickness (perpendicular to the paper) of $1\frac{1}{2}$ in. If $e = 5$ in. and $P = 600$ lb, find the stresses at the edges at G, neglecting the deflection of the clamp.
Ans. 4,200 psi tension; 3,800 psi compression.

118-11. A 6-in. 5.94-lb standard aluminum channel 10 in. long is used as shown in Fig. 187,II. The 500-lb load is placed $\frac{1}{4}$ in. from the outside edge of the flange. Find the stresses in the middle cross section of the web.
Ans. 3,100 psi compression; 2,900 psi tension.

119. Maximum Eccentricity without Reversing Stress. A brick pier laid in lime mortar has no tensile strength, and the tensile strength of masonry laid in cement mortar is uncertain. For this reason, the load on a masonry pier or wall should be so placed that the stress over the entire section shall be compressive.

Example

A solid masonry wall has a width b. How far may the resultant load be placed from the middle without having tensile stress on the opposite face?

For a length l of the wall, the section modulus is

$$Z = \frac{l \times b^2}{6} \qquad A = l \times b$$

$$\frac{P}{A} - \frac{Pe}{Z} = 0 \qquad \frac{P}{l \times b} = \frac{6Pe}{l \times b^2}$$

$$e = \frac{b}{6}$$

The resultant load must not be more than one-sixth the breadth from the middle.

Architects and engineers have the rule that *the resultant load on a rectangular pier or wall shall not fall outside the middle third,* unless the material has tensile strength. When the load has an eccentricity of one-sixth the width, the stress in the outer fibers on the side opposite the eccentricity is zero, while the stress on the outer fibers on the same side is twice the average stress. It is desirable, therefore, to have eccentricity as small as possible.

Problems

119-1. What eccentricity of the load on a short solid cylinder of radius a will make the stress zero on one side? *Ans.* $e = a/4$.

119-2. A hollow cylinder of inside radius b and outside radius a is so loaded as to give zero stress on one side. What is the eccentricity? *Ans.* $e = (a^2 + b^2)/4d$.

119-3. A rectangular pier is 18 in. wide from east to west and 60 in. long. It carries a slightly eccentric, uniformly distributed load of 54,000 lb which makes the stress on the west face 40 psi. Find the eccentricity and the stress on the east face. *Ans.* $e = 0.6$ in.

119-4. A hollow rectangular pier is 24 in. wide from east to west and 60 in. long. The walls are 8 in. thick. It carries a load of 100,000 lb, which is applied at the top by means of a rigid capstone. The compressive stress in the east face is 40 psi. Find the eccentricity of the load. *Ans.* $e = 2.91$ in.

For any section, the maximum eccentricity without reversing stress may be computed by Eq. (118.1). Since the fibers under zero stress are on the side of the center of gravity opposite the load, the negative sign is used in the formula.

$$S = 0 = \frac{P}{A} - \frac{Mc}{I}$$

$$\frac{P}{A} = \frac{Mc}{I} = \frac{Pec}{Ar^2}$$

in which r is the radius of gyration of the section with respect to the axis through the center of gravity.

$$e = \frac{r^2}{c}$$

120. Bending Moment Not about Principal Axis. In all the problems of the preceding articles, the resultant load fell on one principal axis, and rotation took place about the other principal axis. When the section is a circle, a square, an equilateral triangle, or any other regular polygon, the moment of inertia is the same for every axis which lies in the plane of the section and passes through its center of gravity, and every such axis is a principal axis. For other sections, when the load does not fall on a principal axis, the axis of rotation is not the line OE of Fig. 188 but is some other line, such as OG, which lies between OE and the axis of minimum moment of inertia.

To find the bending stress caused by an eccentric load, the eccentricity is resolved into two components parallel to the two principal axes of inertia, and the stress at any point is calculated separately for each component. In Fig. 188, the load perpendicular to the plane of

the paper is applied at the point C. The eccentricity is OC. The components of the eccentricity are x and y. The moment of the load P about the X axis is Py. At any point F, whose coordinates are x_1 and y_1, the bending stress caused by the moment Py is

$$S = \frac{Mv}{I} = \frac{Pyy_1}{I_x}$$

The moment about the Y axis is Px, and the stress at F caused by this moment is Pxx_1/I_y.

The combined direct stress and bending stress at any point (x_1, y_1) when the load is applied at (x, y) is

$$s = \frac{P}{A} + \frac{Pxx_1}{I_y} + \frac{Pyy_1}{I_x} \tag{120.1}$$

The signs of the three terms can usually be determined by inspection from the character of the stress produced, but the student may wish to watch the signs of x, x_1, y, and y_1.

To show that the line OE of Fig. 188 is not the neutral axis of bending, it is helpful to review Arts. 61 to 66. In Fig. 113 the stress distribution of a rectangular section is shown. All loads are applied in a vertical plane perpendicular to

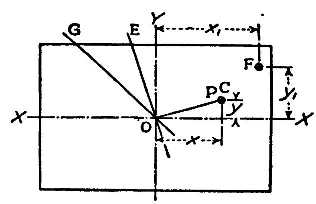


FIG. 188. Resultant load off principal axis.

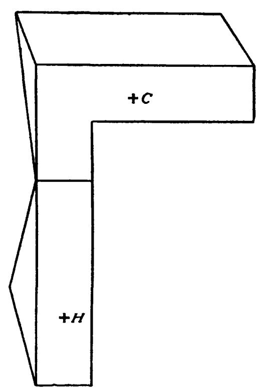


FIG. 189.

the neutral axis BB'. The resisting moment is supplied by the moment of the forces H and C, which form a couple. This resisting couple must lie in the same vertical plane as the plane of the external forces.

Suppose an angle, with one leg vertical, is used as a beam to carry transverse loads. Figure 189 shows the construction of the stress distribution solid on the assumption that the beam bends about the horizontal axis through the center of gravity. The forces H and C, perpendicular to the plane of the paper, form a couple, which is presumed to be the resisting moment. But the plane of this couple is not vertical, and therefore it can be resolved into a couple in a vertical

plane and a couple in a horizontal plane. The couple in the vertical plane will equal the bending moment of the external forces, but since there are no horizontal forces, the horizontal couple will twist the beam laterally. Vertical loads will not bend a beam of this form in a vertical direction. Hence the lateral bending will disturb the neutral axis which was originally assumed (in Fig. 189) horizontal. If there were two equal angles fastened together with their vertical legs back to back, then the combination would be symmetrical with respect to a vertical axis, and the neutral axis for vertical loads would be horizontal.

In Fig. 190, $ABCD$ is a rectangular section with diagonal horizontal. It may be regarded as the end of a cantilever perpendicular to the plane of the paper. If a vertical load P is placed on the end of this cantilever, the deflection will not be vertically downward, but the section will be displaced into a position such as shown in the figure.

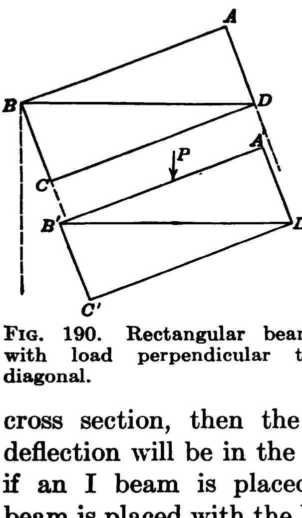


FIG. 190. Rectangular beam with load perpendicular to diagonal.

Figures 188 to 190 are special cases of the general problem in which the bending moment does not lie in the plane of one of the principal moments of inertia. In other words, the bending moment is not about an axis for which the moment of inertia is a maximum or a minimum. If an axis perpendicular to the plane of the bending moment is a principal axis of the cross section, then the beam will bend about this axis, and the deflection will be in the direction of the applied forces. For instance, if an I beam is placed with the web vertical, or a rectangular beam is placed with the long sides of the rectangle vertical, the axis of maximum moment of inertia is horizontal, and a vertical load will deflect the beam vertically downward. If the I beam or rectangular beam were turned 90°, to make the axis of minimum moment of inertia horizontal, the beam again would deflect vertically downward under a vertical load, and the neutral axis would coincide with the principal axis of inertia. When the section of the beam is an equilateral triangle, a square, or any other regular polygon, the moment of inertia is the same for every axis through the center of gravity. Such beams deflect in the plane of the bending moment, no matter what the position of the section may be.

The method of finding the unit stress when the bending moment is not in the plane of a principal axis of inertia is very simple. *Resolve*

the bending moments or the applied forces into two components perpendicular to the two principal axes of inertia, and compute the stress separately for each component. The actual stress at any point is the algebraic sum of the stresses caused by the two components.

Example 1

A cantilever beam of rectangular section is 4 by 3 in. and is 5 ft long. The beam is placed with the 4-in. faces at 30° with the horizontal, and a load of 120 lb is put on the free end. Find the unit stress at each corner, and find the direction of the neutral axis.

The load of 120 lb is resolved into 103.9 lb perpendicular to the 4-in. faces and 60 lb perpendicular to the 3-in. faces. From the first component, the stress in AB and CD of Fig. 191 is

$$S = \frac{103.9 \times 60}{6} = 1{,}039 \text{ psi}$$

From the second component,

$$S = \frac{60 \times 60}{8} = 450 \text{ psi}$$

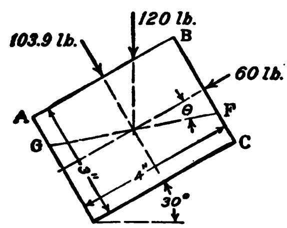


FIG. 191.

At B both stresses are tensile and at D both are compressive. The unit stress at these corners is $1{,}039 + 450 = 1{,}489$ psi. At A the stress caused by the 60-lb component is compressive, while that caused by the other component is tensile. The tensile stress at A and the compressive stress at C are $1{,}039 - 450 = 589$ psi.

The location on the line CB of the point F at which the stress is zero is found by dividing the distance from C to B in the ratio of 589 to 1,489.

$$CF = \frac{589 \times 3}{589 + 1{,}489} = \frac{1{,}767}{2{,}078} = 0.850 \text{ in.}$$

At G on the line AD at a distance of 0.850 in. from A, the unit stress is zero. The line GF through the center of the section is the neutral axis.

The angle θ between the neutral axis GF and a line parallel to the 4-in. faces is given by

$$\tan\theta = \frac{0.65}{2} = 0.325 \qquad \theta = 18°\text{, nearly}$$

The neutral axis makes an angle of 12° with the horizontal.

Example 2

A rectangular block is 12 in. long, measured from east to west, and 10 in. wide, from south to north. It is subjected to a load of 3,600 lb, which is applied 2 in. from the east edge and 2 in. from the north edge. Find the unit stress at each corner (see Fig. 188).

If the east line through the center is taken as the X axis and the north line is taken as the Y axis, $I_x = 1{,}000$ and $I_y = 1{,}440$. The moment about the X axis is 10,800 in.-lb. This moment causes a bending stress of 54 psi at the north

and south edges. The moment about the Y axis is 14,400 in.-lb. This moment causes a bending stress of 60 psi at the east and west edges. The direct compression is 30 psi. The total stresses at the corners are 144 psi compression at the northeast corner, 24 psi compression at the northwest corner, 84 psi tension at the southwest corner, and 36 psi compression at the southeast corner.

Problems

120-1. Solve Example 2 if the load is placed 3 in. from the east edge and 3 in. from the north edge. *Ans.* 111 psi compression at the northeast corner.

120-2. From the results for Example 2, find the location of the points on the south and west edges at which the stress is zero.
Ans. 8.4 in. from the southwest corner on the south edge, and $7\frac{7}{8}$ in. on the west edge.

120-3. A 10- by 6-in. post stands vertical with its 10-in. faces running north and south. A load of 2,400 lb is placed on the top 2 in. from the east face and 2 in. from the north face. A horizontal force of 75 lb toward the west is applied to the east face 24 in. above the bottom, and a horizontal force of 80 lb toward the south is applied to the north face 20 in. above the bottom. Find the unit stress at each corner at the bottom.
Ans. At the northeast corner, compression $= 40 + 72 + 40 - 16 - 30 = 106$ psi.

120-4. A rectangular cantilever, 6 by 10 in., is 6 ft long. The 10-in. faces make an angle of 20° with the vertical. The beam carries a load of 1,200 lb on the free end. Find the unit stress at each corner and the angle which the true neutral axis makes with the horizontal.
Ans. 1,304 psi; 320 psi; 25°16′.

120-5. A 5- by 12-in. rectangular beam, 8 ft long, is supported at the ends with one diagonal of the cross section horizontal. Find the stresses at the corners if the beam carries a uniform load of 650 lb per ft

120-6. A 10-in. 25.4-lb standard steel I beam 10 ft long is used as a cantilever with the web making an angle of 20° with the vertical. Neglecting the weight of the beam, find the stress at the four outside corners when a concentrated load of 244 lb is placed on the end.

120-7. What per cent error is made in the maximum stresses if the weight of the beam in Prob. 120-6 is neglected?

120-8. A 3- by 4-in. rectangular wood beam 5 ft long is used as a cantilever with the 4-in. edges making an angle of 30° with the vertical. Find the allowable uniformly distributed load. *Ans.* 209 lb.

120-9. At what angle with the vertical will a 12-in. 31.8-lb I beam have zero stress at two corners? *Ans.* Arc tan $0.10556 = 6°02′$.

120-10. A 10-in. 15.3-lb standard channel, 12 ft long between supports, carries a load of 400 lb per ft. The web makes an angle of 5° to the right of the vertical upward. The flanges are on the right. Find the unit stress at each corner at the dangerous section.
Ans. Upper left, 8,518 psi compression; upper right, 147 psi compression; lower left, 4,328 psi tension; lower right, 12,699 psi tension.

120-11. A cantilever beam 4 ft long carries a load of 200 lb on the free end. Each section is an equilateral triangle 4 in. on each side. Find the stress at each vertex if the lower base is horizontal. *Ans.* 4,800 psi; 2,400 psi.

120-12. A 6- by 6- by 1-in. standard angle 10 ft long is used as a beam supported at the ends. The angle is placed with legs horizontal and vertical, and a load of 1,000 lb is applied at the middle, over the center of gravity of the section. Find the unit stress at the corners. Here the principal axes are 1–1 for which the moment of inertia is 14.78 and 2–2 for which the

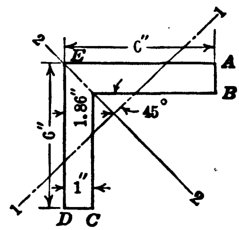


FIG. 192. Angle section.

moment of inertia is 56.14. The bending moment for each axis is $15{,}000\sqrt{2}$.

$$\text{Unit stress at } E = \frac{15{,}000 \times \sqrt{2} \times 1.86 \times \sqrt{2}}{14.78} = 3{,}775 \text{ psi}$$

Ans. Unit tensile stress at $C = 3{,}329 + 1{,}336 = 4{,}665$ psi.

Example 3

A short block of rectangular section is b wide and d thick. Find the eccentricity of the load without reversing stress.

The maximum eccentricity of the load without reversing stress at any given point (x_1, y_1) may be found by

$$\frac{P}{A} + \frac{Pxx_1}{I_y} + \frac{Pyy_1}{I_x} = 0$$

$$\frac{P}{A}\left(1 + \frac{xx_1}{r_y^2} + \frac{yy_1}{r_x^2}\right) = 0$$

$$1 + \frac{xx_1}{r_y^2} + \frac{yy_1}{r_x^2} = 0$$

in which r_x and r_y are the radii of gyration with respect to the X axis and the Y axis, respectively.

To have zero stress at E, at the middle of the right side of Fig. 193,

$$x_1 = \frac{b}{2} \qquad y_1 = 0 \qquad r_y^2 = \frac{b^2}{12}$$

Substituting, this becomes

$$1 + \frac{bx/2}{b^2/12} = 0 \qquad x = -\frac{b}{6}$$

For zero stress at F, at the middle of a side of length b,

$$y = -\frac{d}{2}$$

To find the equation of a straight line upon which a load will give zero stress at the corner C of Fig. 193,

$$x_1 = \frac{b}{2} \qquad y_1 = \frac{d}{2}$$

$$1 + \frac{bx/2}{b^2/12} + \frac{dy/2}{d^2/12} = 0$$

$$-\frac{6x}{b} - \frac{6y}{d} = 1$$

This may represent a straight line with x intercept $-b/6$ and y intercept $-d/2$ (KL of Fig. 193). For the lower right corner G of Fig. 193,

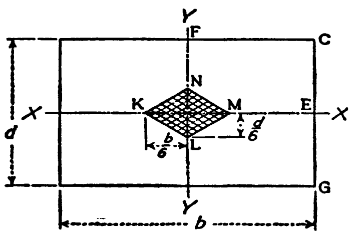


FIG. 193. Area in which load does not reverse stress.

$$x_1 = \frac{b}{2} \qquad y_1 = -\frac{d}{2}$$

and

$$-\frac{6x}{b} + \frac{6y}{d} = 1$$

The line KN of Fig. 193 is a part of the line represented by this equation. A load at any point on this line gives zero stress at G; but if the load were below K, the line of zero stress would fall inside C. The parallelogram $LKNM$ encloses an area anywhere in which a load may be applied without causing reversed stress at the corners or at other points of the rectangular section. This area is sometimes called the *kernel* of the section.

Example 4

Find the kernel of an equilateral triangle (Fig. 194).

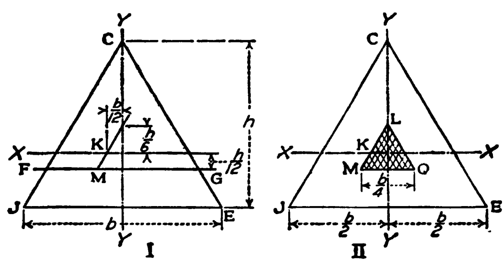


FIG. 194. Kernel for triangular section.

Since this is a regular polygon, the moment of inertia is the same for any axis through the center of gravity. In Fig. 194, the X axis is taken parallel to the base.

$$I = \frac{bh^3}{36} \qquad h = \frac{b\sqrt{3}}{2} \qquad r^2 = \frac{h^2}{18} = \frac{b^2}{24}$$

To find the equation of the line upon which a load will give zero stress at the vertex C, where $x_1 = 0$, and $y_1 = 2h/3$,

$$\frac{2hy}{3} \times \frac{18}{h^2} + 1 = 0 \qquad y = -\frac{h}{12}$$

The load is on the line FG, which is parallel to the base at a distance of one-twelfth the height below the center of gravity. It is evident that the lines for the other two vertices will combine with this one to give an equilateral triangle. However, it is worth while to derive the equation of the line for the vertex E by means of the coordinates of the figure.

$$x_1 = \frac{b}{2} \qquad y_1 = -\frac{h}{3}$$

$$\frac{bx/2}{b^2/24} - \frac{hy/3}{h^2/18} + 1 = 0 \qquad -\frac{12x}{b} + \frac{6y}{h} = 1$$

This represents a straight line which intercepts the X axis at $-b/12$ and the Y axis at $h/6$. Since its slope is $2h/b$, this line is parallel to the side JC of the section. The kernel is an equilateral triangle (Fig. 194,II). Its height is

$$(h/12) + (h/6) = h/4,$$

and its base is $b/4$.

121. Deflection, Moment about a Secondary Axis. When the bending moment applied to a beam is not about a principal axis of the section, it was shown in Art. 120 that the deformation does not take place about an axis perpendicular to the plane of the moment. As a general method, it is necessary to resolve the forces into components perpendicular to the principal axes of the cross section of the beam. The two components of the deflection at any point are calculated separately, and the resultant deflection found from their vector sum. This is analogous to the method of finding bending stresses, except that stresses are added algebraically. An example will illustrate this.

Example 1

Find the deflection of the beam of Example 1 of Art. 120 if $E = 1{,}500{,}000$ psi.

In Fig. 191, the component of 103.92 lb is resisted by the minimum moment of inertia, which is 9 in.^{4} The component of 60 lb. is resisted by the maximum moment of inertia, which is 16 in.^{4} Relatively greater deflection occurs about the axis of minimum moment of inertia.

The component of 103.92 lb causes a deflection of 0.55424 in. downward at 30° to the right of the vertical. The other component causes a deflection of 0.180 in. downward at 60° to the left of the vertical.

$$y = -0.47997 - 0.0900 = -0.56997 \text{ in.}$$
$$x = 0.27712 - 0.15588 = 0.12124 \text{ in.}$$

The beam bends to the right of the vertical about the neutral axis GF.

Example 2

At 3-in. solid shaft, weighing 24 lb per ft, is 10 ft long and is supported at the ends. A pulley weighing 160 lb is 3 ft from the left end and is subjected to a pull of 400 lb 30° below the horizontal in a plane perpendicular to the length of the shaft. Find the deflection at the pulley, if E is 29,000,000 psi.

Resolved vertically, the total vertical load at the pulley is 360 lb. The horizontal pull is 346.4 lb. The deflections at 36 in. from one end are

	In.
From concentrated load of 360 lb............	0.0793
From load of 2 lb per in....................	0.0381
Total vertical deflection..................	0.1174

The horizontal deflection from the load of 346.4 lb is 0.0763 in.

Problems

121-1. In Example 1, find the angle which the resultant deflection makes with the vertical. Compare with the neutral axis of Example 1 of Art. 120.

121-2. A 3- by 10-in. cantilever is 10 ft long. The 10-in. faces are not vertical but make an angle θ with the vertical where $\tan \theta = 3/4$. Find the components of the deflection at the end if the load on the end is 240 lb and $E = 1{,}200{,}000$ psi.

Ans. 3.072 in.; 0.3686 in.; $x = 2.2364$ in.; $y = -2.1381$ in.

121-3. In Prob. 121-2, find the unit stress at each corner at the fixed end. Find the direction of the neutral axis. Compare with the angle which the resultant deflection makes with the vertical.

Ans. 1,612.8 psi tension, 691.2 psi compression at top; 46°18′ with vertical.

121-4. A 4- by 4-in. cantilever is 10 ft long and carries a load of 85 lb at the end. The cosine of the angle which two parallel faces make with the horizontal is $15/17$. Find the components of the deflection if $E = 1{,}000{,}000$ psi.

Ans. $y = -2.295$ in.; $x = 0$. The moment of inertia of any regular polygon is the same in all directions.

121-5. Find the fiber stress at each corner of the beam of Prob. 121-4 at the fixed end. Solve by means of the components.

121-6. A 10-in. 25.4-lb standard I beam is used as a purlin on a roof which makes an angle of 18° with the horizontal, thus making the web of the I beam at an angle of 18° with the vertical. The beam carries a vertical load of 240 lb per ft and a load of 250 lb per ft perpendicular to the roof (and parallel to the web). If the beam is supported at the ends of a 15 ft span, find the unit stress at the corners at the middle and the components of the deflection parallel and perpendicular to the web. $E = 30 \times 10^6$ psi.

Ans. $S = 8{,}344 \pm 6{,}615 = 14{,}959$ psi and 1,729 psi; deflections = 0.1538 in. parallel to web and 0.4222 in. parallel to flange.

121-7. A vertical post 6 in. square and 10 ft long is fixed at the bottom with two faces in the meridian. A horizontal force of 200 lb, south 20° west is applied at the top and a uniform force of 60 lb per ft, directed east, is applied to the west face. Find the unit stress at each corner at the bottom. Find the deflection at the top and 20 in. from the top if the modulus of elasticity E is 1,500,000 psi.

Ans. East component at top = 0.5568 in. $S = 1{,}398$ psi at southeast corner.

122. Miscellaneous Problems

122-1. A short piece of 18-in. 45.8-lb standard steel channel stands on end, and the top is covered by a 4- by 18-in. steel plate which just fits. A load of

10,000 lb is then applied at the center of the plate. Find the stresses at the bottom of the channel (*a*) at the back of the web, and (*b*) at the outside tips of the flanges.

122-2. An 8-in. 11.5-lb standard steel channel is fixed at the base and stands vertically. A 10,000-lb load is applied on the axis of symmetry at a point ¾ in. outside the back of the web. The load is carried by means of an angle bracket which is welded along the two vertical edges with ¼-in. fillet welds. Find the stresses at the four outermost corners of the channel near the base, and the length of fillet welds required.

122-3. In Prob. 122-2, the load may be adjusted to change the ¾-in. position. How far outside the back of the web may the 10,000 lb be placed if the maximum stress at any corner shall not exceed 16,000 psi?

Ans. 0.92 in.

122-4. A 10- by 12-in. rectangular post 3 ft high has its 12-in. edges in the meridian. It carries an 18,000-lb vertical load applied at 4 in. from the north face and 2 in. from the east face. Find the stresses at the four outside corners at the base. Find the stresses at a point 1 in. from the south face and 1 in. from the east face. *Ans.* 30 psi at the northwest corner.

122-5. A 4- by 6-in. vertical wood post 2 ft high has its 6-in. faces in the meridian. A 2-in. standard steel pipe 1 ft high is attached by a flange to the top of the post, making the total height 3 ft. One end of a rope is attached to the top of the pipe, and the other end is fastened to the ground at a point 12 in. north and 9 in. east of the center of the base of the post. The tension in the rope is 2,600 lb. Find the stresses at the four corners of the post at the base. Find the maximum stress in the pipe.

Ans. 23,620 psi in the pipe.

122-6. In Fig. 35 find the stresses at the outside corners of the 4- by 6-in. horizontal member *CD*, at a section *F*, just to the left of the 12,000-lb load.

Ans. 2,687 psi maximum compression.

122-7. A hollow vertical rectangular pier is 8 by 12 in. outside, 4 by 6 in. inside, and 2 ft high. The 12-in. and the 6-in. dimensions are in the meridian. The load is on the east and west center of gravity axis of the cross section. Find the load and the maximum eccentricity so that all parts of the cross section will be in compression, and the maximum stress will not exceed 1,200 psi. *Ans.* P = 600 lb.

122-8. The rectangular post shown in Fig. 195 is set in an inclined position and carries a vertical 3,400-lb load and a horizontal 1,700-lb force. Find the stresses at the corners at the base.

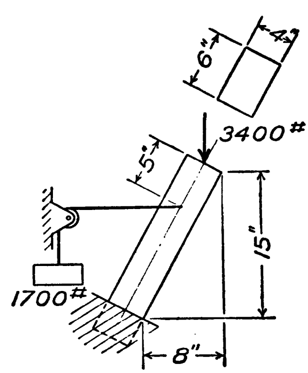


FIG. 195.

Ans. 667 psi compression.

122-9. A solid wall has the resultant load 2 ft from the front edge. The load is 12 tons per running foot. Assuming that the load is so distributed that the top remains plane, find the unit stress in tons per square foot at the front edge if the breadth of the wall is 4, 6, 8, or 10 ft.

Ans. 3 tons per sq ft; 4 tons per sq ft; 3.75 tons per sq ft; 3.36 tons per sq ft compression.

122-10. A solid post in the form of an isosceles triangle, 4 in. wide at the base and 6 in. altitude, carries a load of 4,800 lb on the line of symmetry 3 in. from base. Find the maximum and minimum stress.

Ans. 1,200 psi at the vertex; 0 at the base.

122-11. A 6- by 10-in. beam, 15 ft long between vertical and horizontal supports, carries a load of 240 lb per ft and resists a transverse force of 720 lb 5 ft from one end. Find the maximum stress for each 10-in. interval between the dangerous section for the vertical load and the dangerous section for the horizontal force. *Ans.* 1,200 psi; 1,210 psi; 1,200 psi; 1,170 psi.

122-12. A 20-in. 75-lb standard I beam 24 ft long is supported vertically 3 ft from each end and carries 2,400 lb per ft between the supports. It is held horizontally 1 ft from each end. A force of 3,000 lb at an angle of 60° below the horizontal, in a transverse plane perpendicular to the length of the beam, is applied 9 ft from one end. Find the maximum stress at 10-in. intervals in the region of maximum moment.

Ans. Maximum stress = 18,950 psi.

CHAPTER 13

COLUMNS

123. Definition. In the discussion of eccentric loading in the preceding chapter, no account was taken of the deflection of the body, and of the effect of this deflection upon the eccentricity and the bending stress. Eccentric tension produces a deflection which reduces the eccentricity, as is shown in Fig. 187,III. Eccentric compression, on the other hand, produces a deflection which increases the eccentricity. A yardstick may be placed with one end on the floor and pushed down by the hand at the other end until the middle is bent several inches from the straight line. The original eccentricity, of possibly 0.01 inch, is increased several hundred times, and the bending stress may be sufficient to cause rupture at the middle. If the stick is placed with one end on a platform scale, as shown in Fig. 196, it is found that the load which causes a deflection of 2 inches is little, if any, greater than the load which causes a deflection of 1 inch. While the resisting moment has been doubled, the external moment arm has likewise been nearly doubled. The applied force, therefore, changes very little.

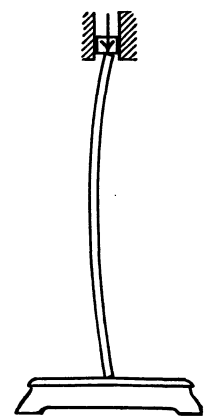


FIG. 196. A long column.

A compression member whose length is several times as great as its smallest transverse dimension is called a *column* or *strut*. Long vertical compression members of buildings and the posts of bridges are usually called columns. The compression members of roof trusses and the vertical compression members of airplanes are called struts. The top chord of a bridge usually acts as a column. The connecting rod of an engine is a column during the forward stroke.

When a column is vertical, the only bending moment is that which is due to the eccentricity of the load and to the deflection. When a column is horizontal or inclined, its own weight applied as in a beam becomes an appreciable factor. The rafters which support a roof act as columns and inclined beams.

Example

A cold-rolled steel rod, 1.250 in. in diameter and 60 in. long, was loaded in compression with cylindrical ends, which were eccentric 0.0202 in. When a load of 6,413 lb was applied, the rod deflected 0.0451 in. at the middle, and when a load of 8,116 lb was applied, the rod deflected 0.1136 in. at the middle. Find the maximum stress in the rod at the middle.

The maximum stress was at the concave side of the beam. The total eccentricity for the first load was 0.0202 + 0.0451 = 0.0653.

$$S = \frac{6{,}413}{1.2272} + \frac{6{,}413 \times 0.0653}{0.19175} = 5{,}226 + 2{,}184 = 7{,}410 \text{ psi}$$

For the second load, $S = 6{,}614 + 5{,}663 = 12{,}277$ pounds per square inch. From this example, it is evident that the maximum unit stress rises much faster than the load, on account of the rapid increase of eccentricity at the middle of the length of the rod.

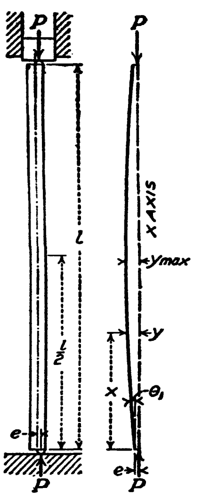


FIG. 197. Eccentrically loaded column.

124. Column Theory. Figure 197 shows a vertical column with ends free to turn about a horizontal axis perpendicular to the plane of the paper. The left figure represents the column as it appears, with no apparent deflection. The right figure represents the central axis of the column with deflections greatly magnified. In order that the beam formulas may apply, the X axis is vertical (parallel to the length of the column) and the Y axis is horizontal and positive toward the left. In this figure, the X axis is on the line of the applied forces, and the origin is at the bottom of the column. The eccentricity, which is the distance of the centers of gravity at the ends from the line of the load, is regarded as positive. In a section at a distance x from the origin, the moment arm is y and the moment is Py. This moment turns the lower end counterclockwise and, therefore, is negative.

From Eq. (76.7),

$$M = EI\frac{d\theta}{dl} \qquad EI\frac{d\theta}{dl} = -Py \tag{124.1}$$

$$EI\frac{d\theta}{dl}\,dy = -Py\,dy \tag{124.2}$$

Since $dy/dl = \sin\theta = \theta$ for small angles,[1]

$$EI\theta\, d\theta = -Py\, dy \tag{124.3}$$

$$\frac{EI\theta^2}{2} = -\frac{Py^2}{2} + C_1 \tag{124.4}$$

Since $\theta = 0$ when $x = l/2$ and $y = y_{\max}$,

$$C_1 = \frac{Py_{\max}^2}{2}$$

$$\theta^2 = \frac{P}{EI}(y_{\max}^2 - y^2) \tag{124.5}$$

If dy/dx is substituted for θ,

$$\frac{dy}{\sqrt{y_{\max}^2 - y^2}} = \sqrt{\frac{P}{EI}}\, dx \tag{124.6}$$

$$\sin^{-1}\frac{y}{y_{\max}} = \sqrt{\frac{P}{EI}}\, x + C_2 \tag{124.7}$$

When $x = l/2$, $y = y_{\max}$,

$$\sin^{-1} 1 = \frac{\pi}{2} = \sqrt{\frac{P}{EI}}\frac{l}{2} + C_2 \tag{124.8}$$

$$\sin^{-1}\frac{y}{y_{\max}} = \frac{\pi}{2} - \sqrt{\frac{P}{EI}}\left(\frac{l}{2} - x\right) \tag{124.9}$$

$$y = y_{\max} \sin\left[\frac{\pi}{2} - \sqrt{\frac{P}{EI}}\left(\frac{l}{2} - x\right)\right] = y_{\max} \cos\sqrt{\frac{P}{EI}}\left(\frac{l}{2} - x\right) \tag{124.10}$$

When $x = 0$, $y = e$, and

$$e = y_{\max} \cos\sqrt{\frac{P}{EI}}\frac{l}{2} \tag{124.11}$$

$$y_{\max} = e \sec\sqrt{\frac{P}{EI}}\frac{l}{2} = e \sec\sqrt{\frac{Pl^2}{4EI}} \qquad \text{Formula XXI}^2$$

$$y = e \sec\sqrt{\frac{P}{EI}}\frac{l}{2} \cos\sqrt{\frac{P}{EI}}\left(\frac{l}{2} - x\right) \tag{124.12}$$

[1] These approximations are equivalent to those used in deriving Formula XVII of Art. 78.

[2] $\sqrt{Pl^2/4EI}$ is an angle in radians, the secant (or cosine) of which is a numerical quantity involved in the solution of these column problems. *It does not refer to any angle on the figure.*

Example

A 2- by 2-in. timber strut, 5 ft long, is tested as a column with round ends. What is the deflection at the middle under a load of 3,200 lb, if $E = 1{,}500{,}000$ psi and the eccentricity is 0.100 in.?

$$\frac{Pl^2}{4EI} = \frac{3{,}200 \times 60 \times 60 \times 3}{4 \times 4 \times 1{,}500{,}000} = 1.44$$

$$\sqrt{\frac{Pl^2}{4EI}} = 1.2$$

$$y_{\max} = 0.100 \sec 1.2 \text{ rad} = 0.100 \sec 68°45' = 0.276 \text{ in.}$$

$$\text{Deflection} = y_{\max} - e = 0.276 - 0.100 = 0.176 \text{ in.}$$

Problems

124-1. In the example above, what is the bending moment at the middle, and what is the maximum unit stress?
Ans. $M = 883.2$ in.-lb; $S = 800 + 662 = 1{,}462$ psi.

124-2. In the example above, if the load is increased to 3,872 lb, what is the deflection? *Ans.* $y_{\max} = 0.403$ in.; $S = 968 + 1{,}170 = 2{,}138$ psi.

124-3. A timber beam 6 in. square and 10 ft long is used as a strut with round ends and eccentricity of 0.5 in. Find the deflection at the middle under a load of 18,000 lb if $E = 1{,}200{,}000$ psi. Find the maximum and minimum compressive stress at the middle if the eccentricity is parallel to the sides.
Ans. $D = 0.5(\sec 40°31' - 1) = 0.1576$ in.; maximum $S = 829$ psi, minimum $S = 171$ psi.

124-4. Solve Prob. 124-3 for the stress if the eccentricity is parallel to a diagonal of the section. *Ans.* $S = 965$ psi and 35 psi.

124-5. Solve Prob. 124-3 for a load of 23,040 lb. *Ans.* $D = 0.2176$ in.; $S = ?$

124-6. A 15-in. 33.9-lb. standard channel 80 in. long is used as a column to carry a load of 99,000 lb, which is applied through cylindrical bearings in a plane midway between the surfaces of the web. Find the deflection at the niddle and the maximum and minimum stress if $E = 29{,}000{,}000$ psi.
Ans. $D = 0.2713$ in. $S = 18{,}215$ psi compression and 16,646 psi tension, using $Z = 3.2$.

124-7. Solve Prob. 124-6 if the load is 1 in. from the back of the web.

124-8. A 2-in. round steel rod 10 ft long is used as a column with ends free to turn. Find the deflection at the middle and the maximum fiber stress on the concave side when the load is 8,000 lb and the eccentricity is 0.1 in., if E is 30,000,000 psi.
Ans. $y_{\max} = 0.1 \sec 63°21' = 0.2230$ in.; maximum $S_c = 4{,}814$ psi.

124-9. A column with ends free to turn is made of a 2-in. round steel rod for which E is 30,000,000 psi. The length is 5 ft. Find the deflection at the middle and the maximum unit stress for loads of 20,000, 30,000, 50,000, 60,000, and 70,000 lb for eccentricities of 0.01 and 0.1 in.
Ans.

Load	20,000	30,000	50,000	60,000	70,000
Unit stress:					
For $e = 0.01$	6,763	10,340	19,295	32,483	Infinite
For $e = 0.1$	10,337	17,500	49,800	154,000	Infinite

125. Euler's Formula. The answers to Prob. 124-9 show that the stress becomes infinite for a load of 70,000 pounds with an eccentricity of 0.01 inch or with an eccentricity ten times as great. Since the secant of 90° is infinite, Formula XXI shows that any column will deflect without limit and finally fail if the total load and the dimensions are such that, where n is odd,

$$\sqrt{\frac{Pl^2}{4EI}} = \frac{\pi}{2} \text{ or } \frac{n\pi}{2} \tag{125.1}$$

$$P = \frac{\pi^2 EI}{l^2} \tag{125.2}$$

which is the common form of Euler's formula.[1] The unit load is

$$\frac{P}{A} = \frac{\pi^2 E}{(l/r)^2} \qquad \text{Formula XXII}$$

in which r is the radius of gyration of the section with respect to the axis about which it bends.

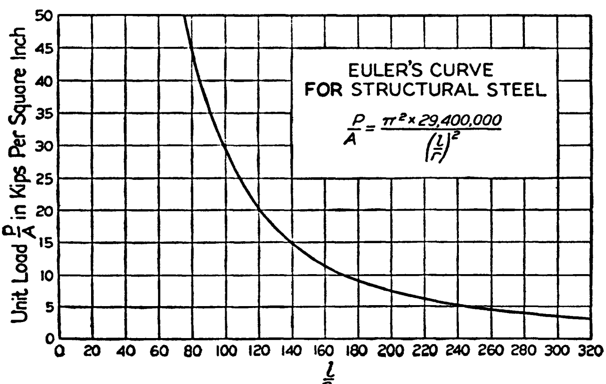


FIG. 198. Euler's curve for a round-end column.

The ratio l/r is called the *slenderness ratio* of the column. In Formula XXII, the slenderness ratio and the *unit* load P/A are the two variables. The formula is of the form $y = a/x^2$, in which a replaces $\pi^2 E$, $y = P/A$, and $x = l/r$. The unit load is a constant divided by the square of the slenderness ratio.

Euler's formula contains the moment of inertia but does not include

[1] Leonhard Euler (1707–1783), eminent Swiss mathematician, astronomer, and advocate of the wave theory of light, first analyzed and published (in 1757) the conditions which cause a long thin homogeneous strut with a perfectly central load to begin to deflect.

the distance to the outer fibers. When the eccentricity is negligible and the slenderness ratio is relatively large, the ultimate load does not depend upon the form of the column, except in so far as the form changes the moment of inertia.

Figure 198 is Euler's curve for a modulus of elasticity of 29,400,000 pounds per square inch. As a mathematical curve, it is infinite in both directions. As an engineering curve, it must not be used above the point at which the unit load is the elastic limit of the material. It will be shown later that it is best not to use Euler's formula for values of the unit load above one-third the elastic limit of the material.

Example

Find the total load with a factor of safety of 2 for a round steel rod 2 in. in diamater if $E = 29{,}400{,}000$ psi and the elastic limit is 30,000 psi, for lengths of 40, 50, 60, and 80 in.

As these lengths are multiples of 10 in., begin with this length and find the others by multiplying by the square of the ratio of 10 to the required length.

Since the moment of inertia of a circle with respect to a diameter is $\pi a^4/4$ and the area is πa^2, the square of the radius of gyration is $a^2/4$, in which a is the radius. The radius of gyration of a 2-in. circle is ½ in. For $l = 10$ in.,

$$\frac{P}{A} = \frac{9.87 \times 29{,}400{,}000}{400} = 725{,}445 \text{ psi}$$

which is many times the ultimate strength of the material.

Length, in.	l/r	Unit load, psi	Total safe load, lb
40	80	45,340	
50	100	29,017	45,580
60	120	20,151	31,652
80	160	11,335	17,805

For the 40-in. length, P/A, as calculated by the formula, is 45,340 psi, which is above the elastic limit of the material. For the 50-in. length, the unit load is nearly the elastic limit. The result may be used with the factor of safety of 2 if it is certain that the eccentricity is very small.

Problems

125-1. In the example, find the unit load for values of the slenderness ratio from 100 to 300 at intervals of 20. Plot.

125-2. When the strut used in the example of Art. 123 was adjusted to small eccentricity (less than 0.0003 in.), the deflection at the middle caused by a load of 9,400 lb was 0.0061 in. When the load was 9,800 lb, the deflection was 0.0104 in. When the load reached 10,020 lb, the deflection was 0.1064

in. When the machine was run still further, the deflection increased with no increase of load. Calculate EI for this cold-rolled steel from Euler's critical load. Calculate E.

Ans. $EI = 3{,}654{,}700$ lb-in.^{2} $E = 30{,}480{,}000$ psi.

125-3. A spruce strut, tested at the Bureau of Standards, was 1.75 in. square and 6 ft 3.75 in. long. The ultimate load was 3,020 lb. Find E by Euler's formula. What was the value of l/r? *Ans.* $E = 2{,}248{,}000$ psi; $l/r = 150$.

125-4. Compression readings for the strut of Prob. 125-3 were taken with a gage length of 30 in. When the load changed from 305 to 2,593 lb, the compression in the gage length was 0.0098. What was the unit stress and what was the unit deformation? Calculate E from these readings. When the load changed from 305 to 2,745 lb, the compression was 0.0107 in. Calculate E. Compare with the results of Prob. 125-3.

125-5. A yardstick with ends rounded was placed vertical, with the lower end on a platform scale and the upper end loaded. When the load was 5 lb, the deflection at the middle was 0.03 in. When the load was 6 lb, the deflection was 0.20 in. When the load was 6.40 lb, the deflection was 1.00 in. The load decreased to 6.28 lb with a deflection of 2.50 in. Calculate EI from the last two readings. *Ans.* 851 lb-in^2; 825 lb-in^2.

125-6. The yardstick of Prob. 125-5, supported as a beam at points 34 in. apart, was deflected $^{31}\!/_{32}$ in. at the middle by a load of 1 lb at the middle. Find EI and compare with Prob. 125-5.

125-7. The yardstick of Prob. 125-6 was 1.08 in. wide and 0.18 in. thick. Find E and I. Find the maximum stress as a beam in Prob. 125-6. Find the maximum stress at the middle for the 6.28-lb load of Prob. 125-5, assuming that the eccentricity was negligible.

125-8. By Euler's formula find the average unit stress for a 5- by 3½- by ½-in. standard steel angle when used as a round-end column 10 ft long, if $E = 30{,}000{,}000$ psi. *Ans.* 11,570 psi.

125-9. Solve Prob. 125-8 if the angle is aluminum. *Ans.* 3,980 psi.

126. Classification of Columns. Columns may be divided, according to the nature of the ends, into the following classes:

I. Both ends free to turn about parallel, horizontal axes, but not free to move laterally (Figs. 196, 197, and 199,I).

II. One end fixed and one end free to turn and free to move laterally (Fig. 199,II).

III. Both ends fixed so that the tangents at the ends do not change (Fig. 199,III).

IV. One end fixed and the other free to turn about one or more horizontal axes, but not free to move laterally (Fig. 199,IV).

Only class I has been considered in the preceding articles.

If L is the entire length of the free portion of a column and l is the length of the cosine curve (sine curve if the origin is shifted) of Fig. 197, $L = l$ for class I.

For class II, the entire length of the column, from the fixed end to

the top, corresponds to one-half of the cosine curve; hence $l = 2L$ in Formulas XXI and XXII.

For class III, the slope is zero at the ends and at the middle. The middle half ABC corresponds to the cosine curve of class I. This portion is represented by l in the column formulas. If L is the entire length DF, then $L = 2l$. A column with both ends *rigidly fixed* will carry as great a load as a column of half its length with ends free to turn. The points A and C of Fig. 199,III are points of inflection (or contraflexure) at which the moment and the curvature change signs.

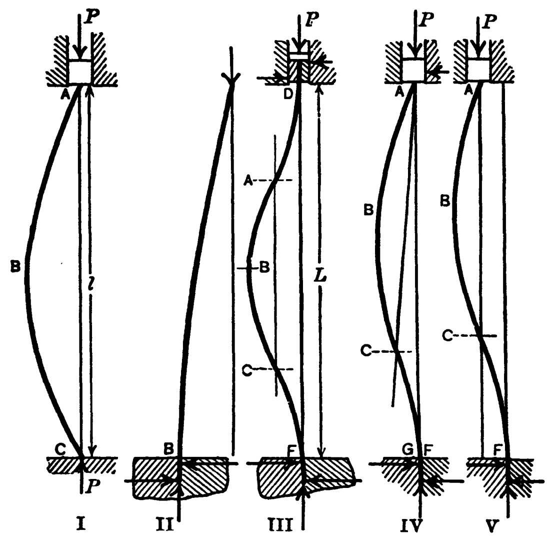


FIG. 199. Types of ideal columns.

The portion AD is equal to one-half of the cosine curve ABC. If the portion AD is rotated 180° about an axis through A perpendicular to the plane of the paper, the point D will fall on B and the curves AB and AD will coincide. The moment is zero at A and C.

The column of class IV is fixed at one end and free to turn at the other, but not free to move laterally. The point of inflection is at C, of Fig. 199,IV. Since the column is free to turn, there is no moment at the top A. As the load is applied, a very small eccentricity at the top, inequality of the material, or slight crookedness causes bending (just as these causes produce bending in classes I, II, and III). Since the moment at the point of inflection C is zero and there are no transverse forces between A and C, the resultant reaction at A is directed from A toward C. The force A, therefore, must have a horizontal

component. The resultant of the horizontal forces at the fixed end is equal and opposite to the horizontal component at A. The portion ABC of Fig. 199,IV forms a cosine curve with the X axis parallel to CA. The lower portion CF forms part of the cosine curve as far as the plane of the body which holds it. Below that plane it is straight. If this portion continued to curve until it became parallel to AC, it would form a complete half of the cosine curve and its length would be equal to AB or BC. Since the portion is vertical at the fixed end, its length is less than one-half of AB and less than one-third of the entire length of the column. The solution of the differential equation shows that AC is nearly $0.7L$. For practical purposes, $l = 0.7L$ and $l^2 = 0.5L^2$, nearly.

It is sometimes stated that l is equal to two-thirds L in a column which is fixed at one end and free to turn at the other. This can be true only under the impractical conditions of Fig. 199,V. In this figure, the top of the column is displaced laterally toward the left. If this displacement is such that the point B is as far from the line AC as the top A is from the vertical line through the fixed end F, then the line AC from the end to the point of contraflexure becomes vertical. In this position, AC is two-thirds of the total length L; there is no horizontal component of the force at the top; and the vertical force is greater than in Fig. 199,IV. The position is unstable. Under a slight vibration the column will deflect to the right of the vertical line through F at the lower end, and the ultimate load will be greatly reduced.

Problems

126-1. A thin yardstick is clamped vertically in a vise at 4 in. from the lower end. When a load of 2 lb is placed on the top, the stick deflects with gradually *increasing* speed and, unless supported or the load removed, finally breaks. Find EI. *Ans.* $EI = 830$ lb-in².

126-2. A yardstick, with ends rounded, was supported and loaded as in Fig. 199,I and was deflected a large amount by a load of 6.1 lb on the top. Find EI by Euler's formula.

126-3. The yardstick of Prob. 126-2 was clamped 4 in. from one end and the load was applied as in Fig. 199,IV. A deflection of 1.5 in. was caused by a load of 15.42 lb. Find EI by Euler's formula.

126-4. The load in Prob. 126-3 was displaced 1 in. south of the vertical line through the bottom. The vertical component of the load when the maximum deflection was 2 in. south was 17.12 lb. Find EI from this experiment.

127. Experimental Check of Theory. Euler's formula and the secant equation, Formula XXI, can be tested best on columns which are free to turn and not free to move laterally at the ends (class I of Fig. 199). A slender column of class II, rigidly clamped at the bottom,

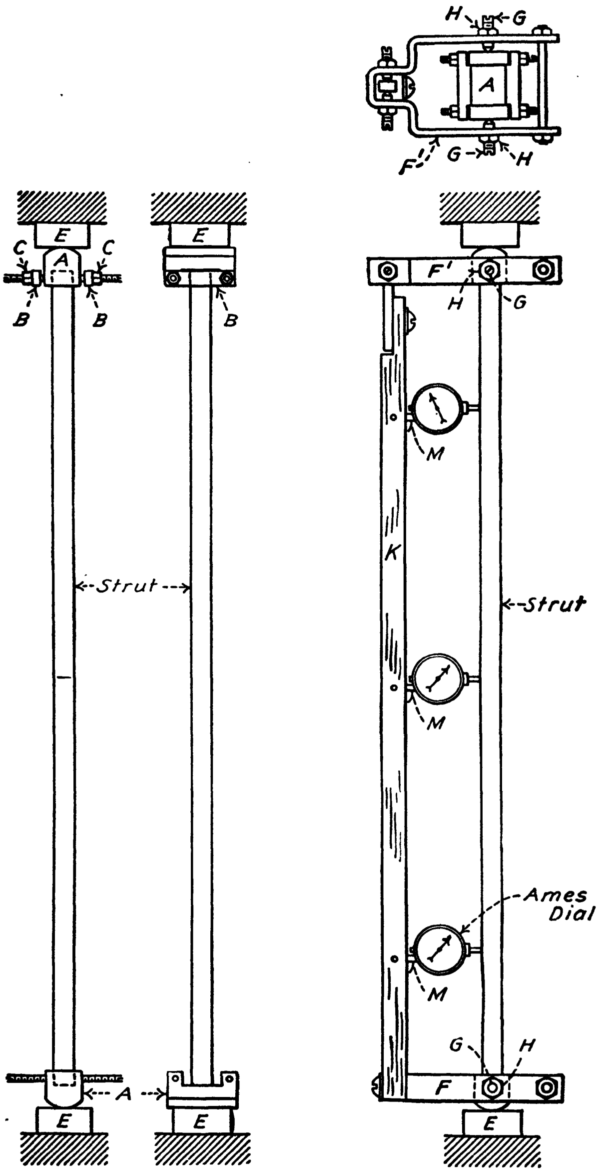


FIG. 200. Deflection apparatus for columns.

may be used for demonstration with an actual load on the top. For accurate results the center of gravity of this load must coincide with the center of the upper end of the column. For stronger struts, the inconvenience of applying actual loads of several tons and the difficulty of fixing the lower end make this form impracticable. Difficulty of fixing the ends and uncertainty as to the eccentricity rule out classes III and IV.

Figure 200 shows the apparatus used by the writer for testing struts of class I.[1] Half cylinders, hardened and ground, are attached to the ends of the column. In the machine these cylinders roll on parallel plates which were casehardened and ground. Clamp bars B (which are not drawn on the lower cylinder) hold the column in place and make it possible to adjust the eccentricity at each end. To measure the deflection and adjust the *eccentricity*, a wooden bar carrying three Ames dials is attached to the heads by means of steel yokes F and F'. These yokes are connected to the centers of the cylinders through cone bearings which permit free rotation around the axis. The wooden bar is rigidly fastened to the lower yoke and connected to the upper yoke through cone bearings.

The column is first adjusted approximately to less than 0.01 inch by measurement. It is then loaded and readings are taken on the upper and lower dials. The eccentricity is shifted under a small load until the deflection readings are the same at both dials. The eccentricity is then shifted equal amounts as read by the dials until a considerable load gives a very small deflection.

The uppermost curve of Fig. 201 shows the deflection of a 1-inch round bar of cold-rolled steel. The deflection at the middle was 0.0002 inch at 9,500 pounds, 0.0011 inch at 10,500 pounds, and 0.0019 inch at 10,800 pounds. At 11,200 pounds the deflection was 0.0168 inch, and at 11,250 pounds it reached 0.1789 inch, far beyond the limits of the drawing. The column was then pushed toward the left by a slight horizontal force applied at the middle with one finger. The beam balanced at 11,250 pounds and the deflection of the middle dial was -0.163 inch.

Slight crookedness may cause two of the dials to read in opposite directions at the start and it is impossible at first to predict which way the strut will finally bend.

With zero eccentricity known, it is then possible to move the column to get any desired eccentricity. Since there must be some initial load to hold the column in place in the testing machine, allowance must be

[1] *Ohio State Univ. Eng. Exp. Sta. Bull.* 25, pp. 11 and 12.

made for the initial deflection. The lower curve of Fig. 201 was obtained by nearly equal positive and negative eccentricities, which made it possible to eliminate the readings at the initial load.[1]

Figure 201 also gives deflection curves for eccentricity of 0.0020 and 0.0100 inch. These were calculated by Formula XXI from the measurements of the test column and the modulus of elasticity obtained by compression test as a square-end column. The points marked by

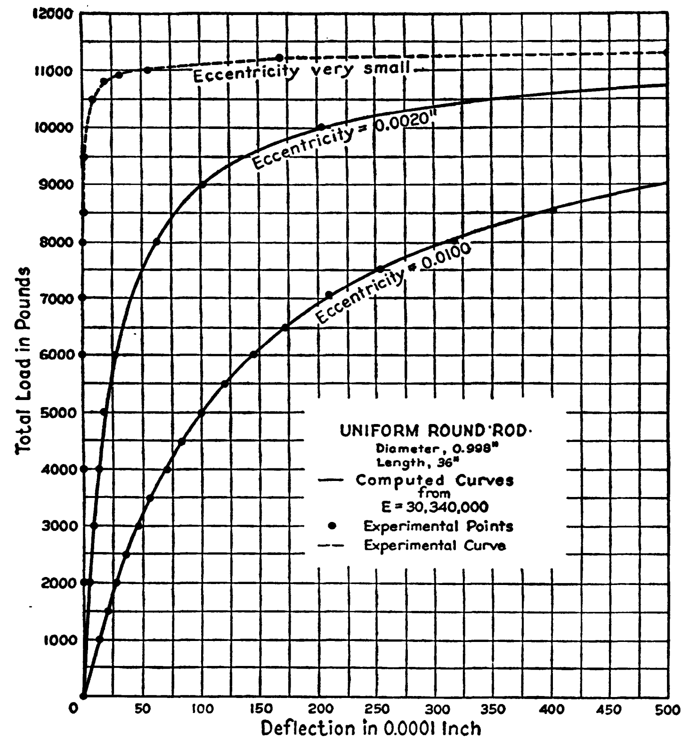


FIG. 201. Load-deflection curves for three eccentricities.

solid circles represent the experimental deflections. The close agreement of these and other tests fully verifies the equations.

A committee of the ASCE, under the chairmanship of Dean Emeritus F. E. Turneaure, undertook a thorough study of columns.[2] The experiments, which were made by Dean M. O. Withey at the University of Wisconsin, included single shapes of structural steel and various

[1] *Ohio State Univ. Eng. Exp. Sta. Bull.* 25, p. 18.

[2] Final Report of the Special Committee on Steel Column Research, *Trans. ASCE*, Vol. 59, No. 98, p. 1376, 1933.

forms of built-up columns. The proportional limit and yield point in both tension and compression and the ultimate strength in tension were measured on specimens taken from the material of each column. Since it had been discovered that there is a great difference in material taken from different parts of a rolled section, sufficient test pieces were taken to investigate these variations. For instance, from a 10-inch 20-pound channel, four test pieces from the toe and root gave 28.9 kips per square inch as the proportional limit and 38 kips per square inch as the yield point; four pieces from the root and web gave 26.6 and 35.8, respectively; two pieces from the root gave 25.3 and 34.6; and two pieces from the web gave 28.0 and 36.8 kips per square inch. The thinner web and toe, which had been subjected to more work in rolling than the root, gave higher values for the proportional limit and yield point. The completeness of the data concerning the properties of the material greatly enhances the value of this investigation.

These columns were tested on roller end bearings designed to support loads of more than 300 tons, and to rotate with relatively small friction. Each consists of a nest of rollers which bear on a curved surface having its axis of curvature at the center of the bearing plate.[1]

The test columns were adjusted to small eccentricity and then moved to give an eccentricity such that $ec/r^2 = 0.5$, or other convenient values of the eccentric ratio.

128. Application of the Secant Formula. Formula XXI of Art. 124 and Eq. (117.1) together give the unit stress in a round-end column under a known load. However, when it is necessary to design or select a column to carry a given load, these formulas are not convenient, since neither the total load nor the unit stress is explicitly given. A problem of this kind must be solved by the method of trial and error.

When a number of columns are to be designed, it is a great saving of time to represent these formulas by means of a table or curve.[2] From Formula XXI,

$$\text{Maximum moment} = eP \sec \sqrt{\frac{Pl^2}{4EI}} \tag{128.1}$$

$$\text{Maximum unit stress} = S_u = \frac{P}{A} + \frac{ePc}{I} \sec \sqrt{\frac{Pl^2}{4EI}} \tag{128.2}$$

$$S_u = \frac{P}{A}\left(1 + \frac{ec}{r^2} \sec \sqrt{\frac{P}{AE}}\frac{l}{2r}\right) \tag{128.3}$$

[1] *Trans. ASCE*, Vol. 95, p. 1131, 1931.

[2] For example, see David H. Crater, A Graphical Solution of the Secant Column Formula, *Civil Engineering*, Vol. 15, No. 9, p. 430, September, 1945.

in which r is the radius of gyration of the column, and l/r is the *slenderness ratio.*

If n represents the factor of safety and S_y the yield point of the material, then the formula for design may be expressed as

$$S_y = \frac{nP}{A}\left(1 + \frac{ec}{r^2}\sec\sqrt{\frac{nP}{AE}}\frac{l}{2r}\right) \tag{128.4}$$

Heretofore the working stress has been obtained by dividing the yield point stress by the factor of safety. In the above equation P/A is the average working stress, which, when multiplied by n, gives S_y, the stress causing elastic failure. It is necessary to introduce n into the equation in this manner because the stress increases faster than the load. The student will often see the above equation with kL instead of l, where k is the factor to correct for different end conditions.

End condition	k
Theoretical fixed ends	0.50
Commercial riveted ends	0.75
Commercial pinned ends	0.875
Theoretical pinned ends	1.0

To determine the relation of P/A to l/r when the unit stress at the concave surface is the ultimate strength of the material, Eq. (128.3) may be written

$$\frac{ec}{r^2}\sec\sqrt{\frac{P}{AE}}\frac{l}{2r} = \frac{S_u}{P/A} - 1 \tag{128.5}$$

It is difficult to solve for P/A in terms of the slenderness ratio but it is easy to solve for slenderness ratio in terms of the unit load.

The yield point is taken as the ultimate unit load for structural steel. While very short columns may be stressed beyond this point without failure, a permanent deformation is not desirable. The average yield point for structural steel is about 36,000 pounds per square inch. The ASCE Column Committee has adopted a lower value, which is called the *useful limit point* (abbreviated ULP). The useful limit point has been defined as that point on the stress-strain diagram of a centrally loaded column at which the slope of the tangent is one-half of the slope of the straight portion. The committee adopted the value of 32,000 pounds per square inch, which is designated as the yield point in secant equations. In the tables which follow, 32,000 pounds will be used as the yield point; 29,400,000 pounds per square inch as the modulus of elasticity; and $0.25 = ec/r^2$ to make the calculations comparable with the work of the committee.

Example

Using the constants of the ASCE committee, calculate l/r for structural steel for unit load of 10,000 psi.

$$\frac{32{,}000}{10{,}000} - 1 = 2.2 = 0.25 \sec \sqrt{\frac{P}{AE}}\,\frac{l}{2r}$$

$$\sec \sqrt{\frac{10{,}000}{29{,}400{,}000}}\,\frac{l}{2r} = 8.8$$

$$\log \sec \sqrt{\frac{1}{2{,}940}}\,\frac{l}{2r} = 0.94448$$

$$\sqrt{\frac{1}{2{,}940}}\,\frac{l}{2r} = 83°29' = 1.45705 \text{ rad}$$

$$\frac{l}{r} = \sqrt{2{,}940} \times 2 \times 1.45705 = 158.0$$

Table 15 gives the values of l/r for a series of values of P/A.

TABLE 15. ULTIMATE UNIT LOAD ON A STEEL COLUMN WITH ROUND ENDS
$E = 29{,}400{,}000$ $S_u = 32{,}000$ $ec/r^2 = 0.25$

$\frac{P}{A}$	$\frac{S_u}{\frac{P}{A}} - 1$	$\sec \sqrt{\frac{P}{AE}}\,\frac{l}{2r}$	Log of $\sec \sqrt{\frac{P}{AE}}\,\frac{l}{2r}$	$\sqrt{\frac{P}{AE}}\,\frac{l}{2r}$		$\frac{l}{r}$
				Degrees	Radians	
2,000	15	60	1.77815	89°03′	1.5542	376.9
4,000	7	28	1.44716	87°57′	1.5350	263.2
6,000	$\frac{13}{3}$	$\frac{52}{3}$	1.23888	86°42′	1.5132	211.8
8,000	3	12	1.07918	85°13′	1.4873	180.3
10,000	2.2	8.8	0.94448	83°29′	1.4571	158.0
12,000	$\frac{5}{3}$	$\frac{20}{3}$	0.82391	81°22′	1.4201	140.6
14,000	$\frac{9}{7}$	$\frac{36}{7}$	0.71120	78°47′	1.3750	126.0
16,000	1	4	0.60206	75°31′	1.3180	113.0
18,000	$\frac{7}{9}$	$\frac{28}{9}$	0.49292	71°15′	1.2435	100.5
20,000	0.6	2.4	0.38021	65°23′	1.1412	87.5
21,000	$\frac{11}{21}$	$\frac{44}{21}$	0.32123	61°30′	1.0734	80.3
22,000	$\frac{5}{11}$	$\frac{20}{11}$	0.25964	56°38′	0.9884	72.3
23,000	$\frac{9}{23}$	$\frac{36}{23}$	0.19457	50°17′	0.8776	62.8
24,000	$\frac{1}{3}$	$\frac{4}{3}$	0.12494	41°25′	0.7229	50.6
25,000	0.28	1.12	0.04922	26°46′	0.4672	32.0

Figure 202 is a graph of Table 15 with P/A as ordinates and l/r as abscissas. The figure includes Euler's curve for $E = 29{,}400{,}000$ pounds per square inch.

Columns with relatively large slenderness ratio fail by bending.

The uppermost curve of Fig. 201 illustrates the behavior of a slender column when the eccentricity is extremely small and the ends are free to turn with the minimum friction. The deflection is very small until the load approaches Euler's critical value. When the critical load is reached, the deflection continues to increase with no increase of load. If there had been a live load on the column, which would follow up at its full value, the column would have continued to bend indefinitely until completely ruined.

With some friction at the compression heads, the load may exceed the critical value with very little deflection. However, when the slight deflection produces sufficient moment, the heads turn suddenly, a large

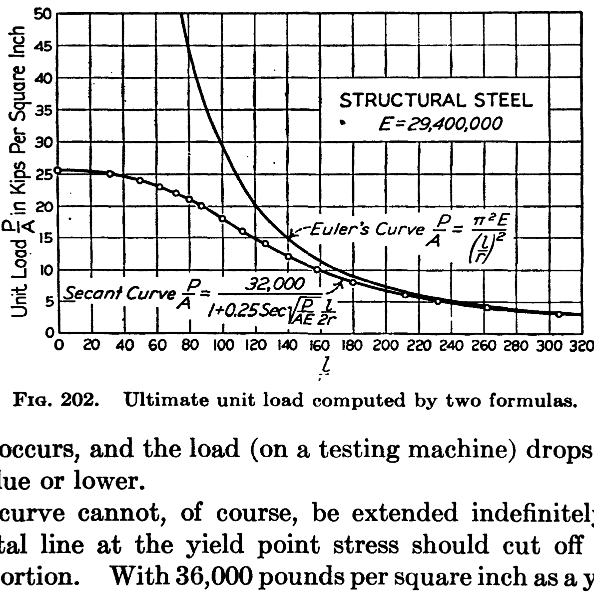


FIG. 202. Ultimate unit load computed by two formulas.

deflection occurs, and the load (on a testing machine) drops to Euler's critical value or lower.

Euler's curve cannot, of course, be extended indefinitely upward. A horizontal line at the yield point stress should cut off the upper unusable portion. With 36,000 pounds per square inch as a yield point, any length with a slenderness ratio less than 95 will not follow Euler's formula but will be governed, to all practical purposes, by the critical stress of the yield point.

The expression ec/r^2 which occurs in the secant formula will be called the *eccentric ratio*. If the numerical value for it is known, the secant formula may be used to find the average unit stress. However, its value is seldom known precisely because columns usually have some initial crookedness and because it is not possible to place the load exactly where it is intended to be, even in laboratory experiments. To show the effect of the eccentric ratio on the stress, the curves of Fig. 203 have been drawn. The curves of ultimate strength from these

secant formulas approach Euler's curve as a limit. Instead of the real yield point, the useful limit point adopted by the ASCE Column Committee has been used.

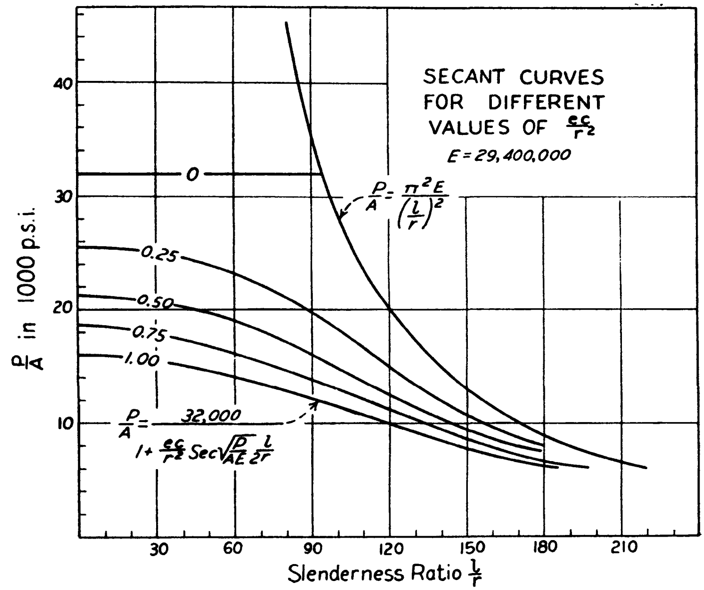


FIG. 203.

Problems

128-1. Find ec/r^2 for a 6- by 8-in. rectangular section if the resultant load is midway between the 6-in. sides at 2.8 in. from the nearest 8-in. side.

Ans. $ec/r^2 = 0.2$.

128-2. Find ec/r^2 for a circular section 3 in. in diameter when the eccentricity is $\frac{1}{6}$ in.

128-3. Find the eccentric ratio for the hard-steel rod column which, when tested, had a load-deflection curve as shown by the bottom curve of Fig. 201.

129. End Conditions in Actual Columns. The classification of columns in the preceding article represents ideal conditions, which are only approximated in practice. The columns in actual use are:

Round-end columns, which end with spherical or cylindrical surfaces. They sometimes end with knife-edges, which may be regarded as cylinders of small radii. The round surfaces roll on plane surfaces with practically no friction. Round-end columns are not used in structures and are rarely used in machines. Since they meet very closely the conditions of class I with ends free to turn, they are frequently used in tests to check the accuracy of theory.

Pin-end or hinged-end columns end with cylindrical surfaces which turn in *cylindrical bearings* (Fig. 204,I). Figure 204,II shows one end of a pin-connected column made of two channels latticed together. This form of connection is commonly used in bridges. A column which

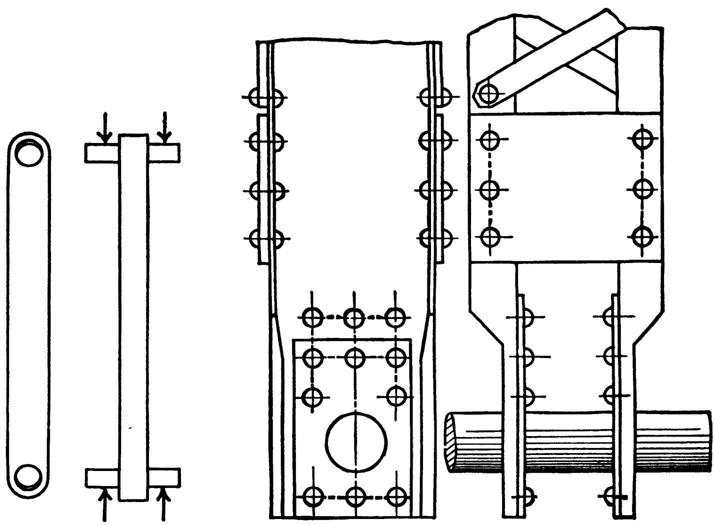


FIG. 204. Pin-end columns.

ends with a ball and socket is practically the same as a hinged-end column, except that it is free to turn in any plane instead of in the single plane normal to the axis of the hinge.

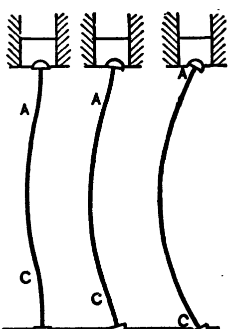


FIG. 205. Deflection of hinged-end column.

If the pin of a hinged-end column rolled on a smooth plane surface, there would be little friction, and the conditions would be those of the deal round-end column. Usually the pin turns in a closely fitting seat or bearing, which may introduce considerable friction. If the pin is small, the moment arm of the friction is small and there is little resistance to rotation at the end of the column. If the pin is large, there is considerable resisting moment, and the column behaves at first approximately as a column with fixed ends. Figure 205 shows diagrammatically three stages of the deflection of a pin-end column.

For a series of tests at the Watertown Arsenal in 1909, built I columns were made of one 10- by $\frac{3}{8}$-inch plate and four 4- by 3- by $\frac{3}{8}$-inch

angles. The least radius of gyration was 1.65 inches. The pin-end columns of this series were tested with 3-inch pins which rested in $3\frac{1}{64}$-inch seats. The tests were made on a horizontal compression machine with the axis of each pin vertical and parallel to the 10-inch plate.

Tests were made with slenderness ratios at intervals of 25 from 25 to 175, inclusive. Table 16 gives the summary of the tests and Fig. 206

TABLE 16. COMPARATIVE TESTS OF PIN-END AND SQUARE-END COLUMNS (Watertown Arsenal Test of Build I Column.)

Slenderness ratio, $\frac{l}{r}$	Ultimate load, pounds per square inch			
	Square ends		Pin ends	
	Separate columns	Average	Separate columns	Average
25	37,450 36,000 35,580	36,343	37,670 37,870 36,720	37,420
50	34,460 34,560 34,750	34,590	33,800 33,640 34,080	33,840
75	34,690 34,740 34,420	34,617	32,270 32,000 32,110	32,160
100	31,670 32,800 32,800	32,423	31,940 31,950 33,070	32,320
125	29,930 31,300 28,880	30,037	30,000 28,850 28,740	29,197
150	30,300 30,080 30,520	30,300	27,400 28,310 29,190	28,320
175	24,730 26,650 26,720	25,033	13,130 27,010 23,200	21,110

All square-end columns failed by triple flexure. All pin-end columns of slenderness ratio 25 and 50 failed by triple flexure with buckling of the flanges. All pin-end columns with slenderness ratio from 100 to 175, inclusive, failed by sudden springing laterally.

shows them plotted. While these tests may seem out of date, they were carefully controlled, and the results are still valid.

Square-end or flat-end columns end with plane surfaces in contact with plane surfaces. The ends must be accurately fitted to avoid eccentricity. If a beam which rests on a square-end column bends under a load, as shown in Fig. 207,II, the load on the column becomes eccentric. Footings which support columns often settle unevenly and cause large eccentricity. Pin-end columns are square ended in the direction of the axis of the pin. The pin-end column of Fig. 206 was horizontal in the testing machine with the pins vertical. The maxi-

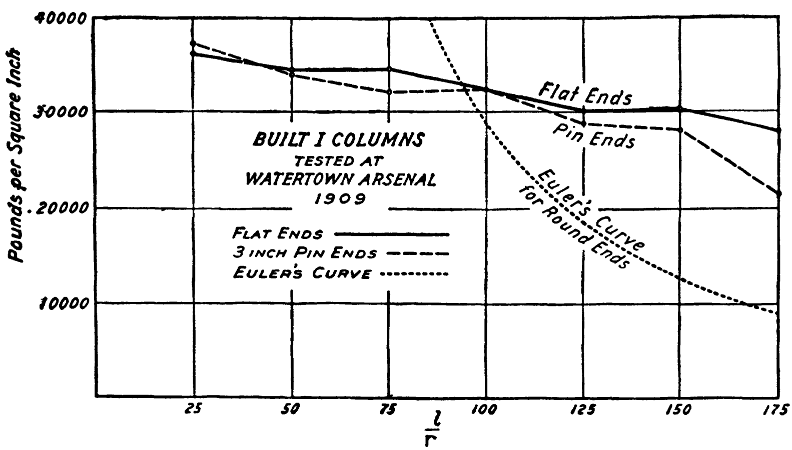


FIG. 206.

mum vertical deflection was only 0.04 inch, while the maximum horizontal deflection recorded was six times as great.

Fixed-end columns may be riveted, bolted, or welded to fixed footings, or to other members of a bridge or building. In a machine, a fixed-end column may be fastened in these ways to the frame or may be cast or forged continuous with it. Since the connection cannot be absolutely rigid and since the member to which the column is "fixed" must suffer some distortion, the tangent at the end of the column does not remain entirely stationary and the conditions of class III are never completely satisfied. If the column is very flexible in comparison with the body to which it is fixed, the ideal case may be approximated and one-half the total length of the column may be used for l in the formulas. In most practical columns, this assumption would introduce a dangerous error.

A column with a pin connection at one end and a square or fixed connection at the other is called a *pin-and-square column*. This column approximates the conditions of class IV of Fig. 199. The yardstick of Prob. 126-3 shows the agreement of experiment with theory. In this experiment, the column was rigidly clamped to a 2- by 4-inch post and was relatively flexible. (The slenderness ratio was more than 800.) A column of ordinary slenderness fastened to a structure of comparable dimensions would not meet so closely the conditions of the theory, and the experimental and calculated results would not agree so well.

Table 16 also gives the results of tests of square-end columns which had the same cross section as those tested as pin ends. The curves of Fig. 206 afford a comparison of hinged-end and square-end columns. Except for the slenderness ratio of 175, the results are much alike. One pin-end column of this last ratio had an ultimate strength of about half as much as the others, which greatly lowers the average for that length. Figure 206 gives Euler's curve for a round-end column with $E = 29{,}000{,}000$ pounds per square inch. For slenderness ratios of 100 and over, the strength of a pin-end column and of a square-end column are considerably above Euler's value. For smaller slenderness ratios the ultimate strength is close to the yield point of the steel. It must be remembered, however, that these test columns were carefully made and were tested with small eccentricity. Moreover, the compression heads of this testing machine are very rigid, so that the square ends of the columns are not free to turn.

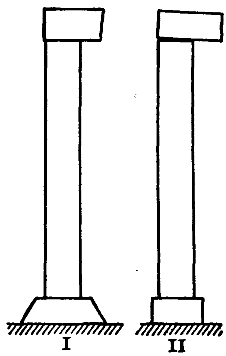


FIG. 207. Square-end columns.

Valuable tests of columns were made by James Christie, in 1883, at the Pencoyd Iron Works. For some of these tests, the so-called *hinged-end columns* were fitted with hemispherical balls turning in sockets. These balls were fitted as accurately as possible by measurement. The final adjustment was made in the testing machine. A small load was applied and the deflection measured. The hemispheres were shifted and the measurement repeated until a considerable load caused no appreciable deflection. The column was then loaded to failure. Since these ball-and-socket joints were lubricated, the friction was smaller than that of a hinged connection of a truss, but considerably more than that of the half cylinders which rolled on plates or in a nest of roller bearings. The ball-and-socket joint gives the strut opportunity to deflect in any direction, which is sometimes an advantage

but more often a disadvantage for experimental studies. These tests were made on wrought-iron struts.[1]

With these hemispherical joints, "when the point of greatest strength was reached, the behavior of the specimen was peculiar. Under ordinary circum-

TABLE 17. PENCOYD TESTS OF WROUGHT-IRON STRUTS

$\frac{l}{r}$	Average results for angles and tees			
	$\frac{P}{A}$ ultimate unit load, pounds per square inch			
	Round ends	Hinged ends	Flat ends	Fixed ends
20	44,000	46,000	49,000	45,000
40	36,500	40,500	41,000	38,000
60	30,500	36,000	36,500	34,000
80	25,000	31,500	33,500	32,000
100	20,500	28,000	30,250	30,000
120	16,500	24,250	26,500	28,000
140	12,800	20,250	23,250	25,500
160	9,500	16,350	20,500	23,000
180	7,500	12,750	18,000	20,000
200	6,000	10,750	15,250	17,500
220	5,000	8,750	13,000	15,000
240	4,300	7,500	11,500	13,000
260	3,800	6,500	10,250	11,000
280	3,200	5,750	8,750	10,000
300	2,800	5,000	7,350	9,000
320	2,500	4,500	5,750	8,000
340	2,100	4,000	4,650	7,000
360	1,900	3,500	3,900	6,500
380	1,700	3,000	3,350	5,800
400	1,500	2,500	2,950	5,200
420	1,300	2,250	2,500	4,800
440	......	2,100	2,200	4,300
460	......	1,900	2,000	3,800
480	......	1,700	1,900	

stances the bar, while bending under the strain, rotated from the start on its hinged ends. When correctly centered, no such rotation occurred at the beginning of the deflection, but the bar bent like a flat-ended strut, till the point of failure was reached, when it rotated on its ends suddenly, as some-

[1] *Trans. ASCE*, pp. 85–122, 1883.

times to spring from the machine. These results could not be secured when the balls or pins rolled on plane surfaces, and were difficult to get when the pins were small."

The effect of the size of the pin was shown in these experiments. Two angles of the same length were cut from the same bar. One of these tested with a 2-inch ball and socket failed at 36,500 pounds per square inch; the other tested with a 1-inch ball and socket failed at 24,010 pounds per square inch.

These and other tests showed how the friction at the ends of a hinged-end column partly fixes the ends and greatly increases the strength. It is a question, however, how much of this is lost on a railroad bridge on account of the vibration of moving trains.

Table 17 gives the results of one series of tests which were made by the Pencoyd Company on rolled angles and tees of wrought iron. The hinged ends were ball-and-socket joints and the round ends were balls on plane surfaces. The columns could bend equally in any direction. It was found that failure always took place in the direction of the least radius of gyration. The figures of Table 17 give some idea of the relative values of the ultimate loads *under the conditions of these experiments.* For a unit load of 25,000 pounds per square inch, for instance, l/r is 80 for round ends, 129 for flat ends (by interpolation between 26,500 and 23,250), 119 for hinged ends, and 144 for fixed ends.

Problems

129-1. Take $l/r = 60$ for round-end columns in Table 17. Find the lengths for hinged, flat, and fixed ends.

129-2. Solve Prob. 129-1 for $l/r = 100$ for round ends. *Ans.* 138; 160; 177.

If all the values for round ends from 40 to 200, inclusive, are taken, the corresponding values of l/r which give the same unit load for the other end conditions are

	Hinged	Flat	Fixed
Maximum	1.45	1.69	1.87
Minimum	1.29	1.50	1.27
Mean of all	1.37	1.60	1.72

Only one value fell below 1.50 for fixed ends.

As far as these tests go, they indicate that a flat-end column 16 feet long, a fixed-end column 17.2 feet long, or a hinged-end column

13.7 feet long will carry the same load as a round-end column 10 feet long of the same cross section.

These results converted to effective length kL give values of k

Round	Hinged	Flat	Fixed
1.0	0.73	0.62	0.58

The above should be compared with the suggestions of practice in Art. 128.

The conditions in actual structures may be very different from those of these carefully conducted experiments. Columns may be fixed to rather flexible beams. The eccentricity is likely to be greater than in these tests, and other factors may reduce the advantage which a hinged-end, flat-end, or fixed-end column has over a round-end column.

CHAPTER 14

WORKING FORMULAS FOR COLUMNS

130. Kinds of Formulas. The secant formulas of Art. 124 are theoretically correct within the limits of the assumptions of beam theory, which are universally accepted. Careful tests, *in which the conditions of the theory are fully met,* amply verify these formulas. For round-end columns of uniform material, the results of experiments agree with theory within the limits with which the modulus of elasticity can be determined by direct compression. Moreover, the deflections at any point on a column agree with Eq. (124.12). Euler's formula, which is a special case of the secant formula, also agrees with experiments.

Euler's formula with a factor of safety may be used as a working equation for large slenderness ratios.

The secant formula applies to all lengths. It approaches Euler's formula as a limit for large slenderness ratios. Mathematically it is difficult to use unless curves are plotted similar to that of Fig. 203. On account of this difficulty, most engineers prefer some more convenient approximate equation. Such equations may be made which lie well inside the limits of the uncertainty as to the amount of eccentricity.

Engineering practice has adopted straight line, parabolic, and Rankine type equations for most design work.[1] Some of these equations have been founded on experiment, often inadequate, some on approximate theory, and a few on accurate theory. The Column Committee of the ASCE has made the most valuable set of tests of full-size columns and has studied these tests with the secant formulas. Moreover, on the basis of all previous tests and a knowledge of the conditions in structures, the committee, as a piece of mature engineering judgment, has recommended working equations in the form of parabolas which agree closely with the secant equation over the entire range of slenderness ratios used in primary columns.

131. Fixed-end Structural-steel Columns. Parabolic Type Formula. The ASCE committee recommends that three-fourths the length

[1] The Specifications for Steel Railway Bridges of the American Railway Engineering Association require the secant equation with an eccentric ratio of 0.25 for compression members having slenderness ratios greater than 140, and for compression members of known eccentricity.

of a riveted column be used for l in the secant formula. The committee adopted a useful limit point of 32,000 pounds per square inch and an eccentric ratio of 0.25 for the secant formula which best fitted the tests. In Fig. 208 this curve has been plotted using hollow circles to represent

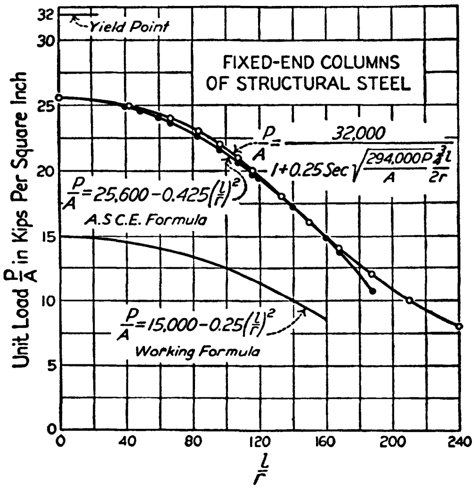


FIG. 208. Ultimate and allowable unit loads.

the calculated points. For fixed-end columns, for ultimate unit loads, the committee recommends the parabola

$$\frac{P}{A} = 25{,}600 - 0.425\left(\frac{l}{r}\right)^2 \qquad (131.1)$$

The maximum deviation from the secant curve is less than 2 per cent up to a slenderness ratio of 160. For structural steels used for bridges where the working tensile stress is 18,000 pounds per square inch, the following working formula for compression members with riveted ends is recommended:

$$\frac{P}{A} = 15{,}000 - 0.25\left(\frac{l}{r}\right)^2 \qquad \text{Formula XXIII}^{1}$$

A comparison of Eq. (131.1) with Formula XXIII indicates that the factor of safety of the working formula is 1.7, but it must be remembered that the unit stress has already been reduced from 32,000 to 25,600 pounds per square inch to allow for initial eccentricity and

[1] Equations of this form are frequently called *Johnson's parabolic equations* after the late J. B. Johnson, who first proposed them and who made many valuable contributions to experimental and applied strength of materials.

crookedness. Both Eq. (131.1) and Formula XXIII are shown in Fig. 208.

Problems

131-1. Find the unit safe load and the total load on an 8-in. 18.4-lb standard I beam for lengths of 5 and 8 ft. (Always solve first for P/A and observe the reasonableness of the unit stress. Then multiply by the area.)
Ans. $P = 73{,}300$ lb; 62,600 lb.

131-2. Find the unit load and the total safe load on a 4- by 4- by ½-in. standard angle, 9 ft 9 in. long, as a column with fixed ends.
Ans. $P/A = 9{,}375$ psi; $P = 35{,}156$ lb.

131-3. Find the slenderness ratio and the ultimate load on a 5- by 3.5- by ⅝-in. standard angle as a column with fixed ends, 10 ft long.
Ans. $P = 42{,}300$ lb.

131-4. Find the total safe load that may be carried by a 10- by 8-in. 45-lb wide-flange section 20 ft long. *Ans.* 150,900 lb.

131-5. A plate-and-angle column section is made of a 10- by ⅜-in. web plate and four 5- by 3½- by ⅝-in. angles. Find the total safe load if the length is 20 ft. *Ans.* 285,000 lb.

132. Hinged-end Structural-steel Columns. Parabolic Type Formula. The ASCE Column Committee recommends for hinged-end steel columns, for slenderness ratios up to 160, the equation

$$\frac{P}{A} = 15{,}000 - \frac{1}{3}\left(\frac{l}{r}\right)^2 \qquad \text{Formula XXIV}$$

This formula was obtained from the same theoretical equation by substituting 85 per cent of the length for 75 per cent, which was used in the derivation of the previous case. This seems reasonable enough although some engineers use an effective length of seven-eighths of the length.

The AISC recommends

$$\frac{P}{A} = 17{,}000 - 0.485\left(\frac{l}{r}\right)^2 \qquad (132.1)$$

for axial-loaded steel columns where the slenderness ratio does not exceed 120. (Beyond this point AISC uses a modified Rankine type formula which will be discussed later.) The AISC specifications do not distinguish between riveted ends and pinned ends, probably because building frames are usually riveted or welded. Both Formula XXIV and Eq. (132.1) give reasonable stresses when applied to a problem. The answers differ only in their factor of safety. The ASCE formula has a factor of safety of approximately 1.7 throughout its range, based on the tests of structural columns referred to previously. If the AISC

formula is based on the same tests, its factor of safety varies from 1.5 (for $l/r = 0$) to 1.74 (when $l/r = 120$) for hinged columns. For riveted ends the factor of safety would be slightly more at a slenderness ratio of 120.

Problems

132-1. Solve Prob. 131-1 by both formulas given in this article.

132-2. Solve Prob. 131-2 by both the AISC and the ASCE hinged-ends formula.

132-3. An 8- by 8-in. 31-lb wide-flange section is used as a column 20 ft long with pin ends. The pins are perpendicular to the web at the ends. (Several plates are welded to the web at the ends so that the pins will have sufficient bearing area.) Find the safe unit stress using (*a*) the AISC formula; (*b*) the ASCE hinged-end formula; and (*c*) the ASCE fixed-end formula, assuming that the plates restrain the ends sufficiently to make bending against the pins equivalent to fixed ends. *Ans.* 14,680 psi; 13,400 psi; 11,430 psi.

132-4. A plate-and-channel column 20 ft long is made of two 10-in. 15.3-lb channels, placed 6 in. apart back to back and loaded through pins at right angles to the webs of the channels. The two plates are each 12 by ¼-in. Find the allowable load as a hinged-end column.
Ans. $P/A = 14{,}000$ psi; $P = 207{,}160$ lb.

132-5. The column of Prob. 132-4 may be regarded as fixed with respect to axes perpendicular to the pins. Find the unit load by Formula XXIII and find the total load if necessary. *Ans.* $P/A = 13{,}870$ psi; $P = ?$

133. Euler's Extension of Parabola. The ASCE specifications permit the use of Formulas XXIII and XXIV to $l/r = 160$. For greater slenderness ratios it is desirable to have an extension of the Euler type for each of these. Representing P/A by y and l/r by x, the parabola becomes $y = S_u - kx^2$, in which k is a constant which may or may not be known, and Euler's equation is $y = C/x^2$. If x' and y' are the coordinates of the point of tangency of the two curves, then

$$S_u - kx'^2 = \frac{C}{x'^2} \qquad S_u x'^2 - kx'^4 = C \tag{133.1}$$

At the point of tangency,

$$\frac{dy}{dx} = -2kx' = -\frac{2C}{x'^3} \tag{133.2}$$

$$kx'^4 = C \tag{133.3}$$

Eliminating k,

$$S_u x'^2 = 2C \qquad x'^2 = \frac{S_u}{2k} \tag{133.4}$$

Since $y = C/x^2$,

$$y' = \frac{S_u}{2} \tag{133.5}$$

Example 1

Find the equation of the Euler extension of Formula XXIV.

The ordinate of the point of tangency of the parabola and Euler's curve is one-half the y intercept of the parabola. This is the key to the problem and is given by Eq. (133.5). Substitute 7,500 for y in $y = 15{,}000 - \frac{1}{3}x^2$ and solve for $x = 150$.

The coordinates $(x, y,)$ have the value (7,500, 150) and must also satisfy an Euler equation $y = \pi^2E/nx^2$, where n is the factor of safety of the parabolic equation its point of tengency with Euler's equation. It is not necessary to know n.

$$\frac{\pi^2E}{n} = yx^2 = 168{,}750{,}000$$

The required equation is

$$\frac{P}{A} = \frac{168{,}750{,}000}{(l/r)^2} \tag{133.6}$$

Example 2

Two 4- by 4- by $\frac{5}{8}$-in. angles are welded together to form a tee 18 ft long between pins which are placed at the ends on the center of gravity of the section, perpendicular to the axis of symmetry. What total safe load might this column be expected to carry?

On axis of pin $r = 1.20$	On axis perpendicular to pin $r = 1.72$
Slenderness ratio $= 180$	Slenderness ratio $= 126$

Try ASCE hinged-ends formula $P/A = 15{,}000 - \frac{1}{3}(180)^2 = 4{,}200$ psi. This is too conservative. Whenever the unit stress comes out less than half of 15,000, the point (on this parabola) is beyond the point of tangency with Euler's. (See Example 1 where $x = 150$.) Hence use Eq. (133.6), which gives $P/A = 5{,}210$ psi.

$$P = (9.22)(5{,}210) = 48{,}000 \text{ lb}$$

It is not necessary to check the other axis. Why?

Two questions may arise to vex the student at this point.

1. What load would it be reasonable to expect the column to carry safely under these conditions? This question is answered by a solution similar to the examples above.

2. What load would the Blank Specifications permit on this column? The designer must follow the particular specifications governing his work. If it is a city building code, perhaps it limits the slenderness ratio of columns in the structure to, say, 170. This would prohibit the use of the section used in Example 2 for lengths over 17 ft. Regardless of the load to be carried, it would be necessary to use a section with a larger radius of gyration to comply with the law.

In Fig. 209 the ASCE hinged-end formula with its Euler's extension is plotted. Also shown is the AISC formula.

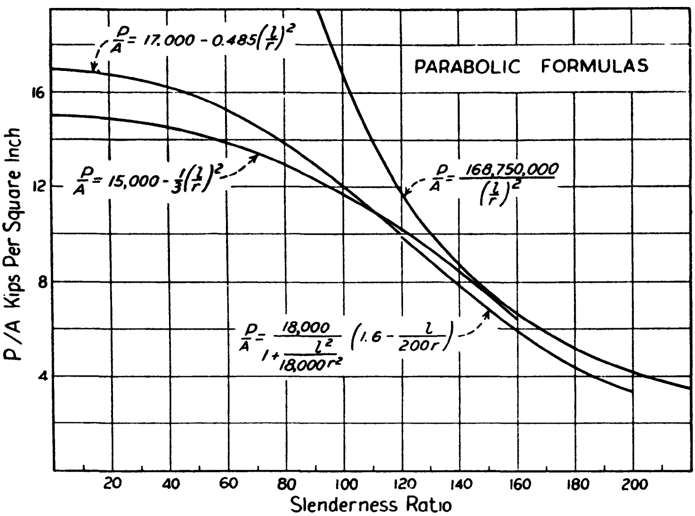


FIG. 209.

Problems

133-1. Derive the equation of Euler's extension for Formula XXIII for a fixed-end steel column, and indicate the slenderness ratio at the point of tangency.

$$Ans.\ \frac{P}{A} = \frac{225,000,000}{(l/r)^2}.$$

This value of 225,000,000 in place of 290,000,000 would seem to represent a small safety factor of 1.3, but it must be remembered that l is the entire length of the fixed-end column. If $\frac{3}{4}l$ be regarded as the free length, the safety factor is increased to 2.2.

133-2. Two 4- by 3- by ½-in. standard steel angles are welded with the long legs back to back to form a T-shaped section. By the ASCE fixed-end formula, find the total safe load for a length of 10 ft. What safe load could it carry if the length were 20 ft? *Ans.* 81,250 lb; 36,600 lb.

133-3. Find the total safe load on an 8- by 5¼-in. 20-lb wide-flange section which is connected by pins at right angles to the web, for lengths of 9 and 18 ft.

133-4. A 3- by 1-in. bar is welded to rigid bodies at each end. Below what length will Formula XXIII apply when the bar is subjected to compression? Find the total safe loads for lengths of 3, 5 and 8 ft.

Ans. 50 in.; 33,340 lb; 15,625 lb;?

133-5. Derive an Euler extension for the AISC formula.

Example 3

Derive a parabolic equation similar to Formula XXIII for a round-end steel column by means of the equations of this article without reference to the secant formula. Also write Euler's extension.

The yield point was 32,000 psi, but with an eccentric ratio of 0.25 this is reduced to 25,600 when the slenderness ratio is zero (see Fig. 203). The ultimate strength

is therefore reduced to 80 per cent of the yield point. The problem is to find k in the equation

$$y = 25{,}600 - kx^2$$

so that the parabola will fit the secant curve (in Fig. 203) which has an eccentric ratio of 0.25.

In Eq. (133.5), y' must be 12,800 psi and this value must fit an Euler equation also. This new Euler equation will have ordinates only 80 per cent of those shown in Fig. 203. In other words, both ends of the secant curve have been moved down (when the eccentric ratio is 0.25) and the Euler curve has been made to fit the right end with a stress of 12,800 psi.

$$y' = 12{,}800 = \frac{P}{A} = \frac{0.80(29{,}400{,}000\pi^2)}{x'^2} \tag{133.7}$$

$$x'^2 = 18{,}506 \qquad x' = \frac{l}{r} = 136$$

$$k = \frac{S_u}{2x'^2} = 0.692 \tag{133.8}$$

$$\frac{P}{A} = 25{,}600 - 0.692\left(\frac{l}{r}\right)^2 \tag{133.9}$$

up to the slenderness ratio of 136.

For a working formula with a factor of safety of 1.7,

$$\frac{P}{A} = 15{,}000 - 0.407\left(\frac{l}{r}\right)^2 \tag{133.10}$$

For slenderness ratios above 136,

$$\frac{P}{A} = \frac{139{,}500{,}000}{(l/r)^2} \tag{133.11}$$

Round-end columns are little used in structures. Slender pin-connected columns in airplanes should be treated as round-end columns. The connecting rod of an engine is a round-end column which is further complicated by transverse forces of the weight and the centrifugal force.

Problems

133-6. On graph paper plot (*a*) the secant curve with an eccentric ratio of 0.25 from Fig. 203; (*b*) Euler's curve with the right member multiplied by 0.8; and (*c*) Eq. (133.9).

133-7. Following the method of Example 3 above, derive a parabolic equation for a round-end steel column to fit the secant equation when the eccentric ratio is 1.0.

133-8. Calculate the safe load that could be placed on a hollow round-end steel column 4 in. outside diameter and 2 in. inside diameter when the length is (*a*) 11 ft and (*b*) 15 ft. Use the formulas of Example 3.

Ans. 88,000 lb; 50,700 lb.

Example 4

Round-end spruce struts tested at the Bureau of Standards[1] had an average modulus of elasticity of 1,910,000 and ultimate strength of 5,200 psi, which was obtained by extending the graph of the columns back to zero slenderness ratio.

Derive a parabolic equation and Euler's extension on the assumption of eccentric ratio = 0.1.

$$\frac{5{,}200}{1.1} = 4{,}730 = S_u \qquad \frac{1{,}910{,}000\pi^2}{1.1} = 17{,}100{,}000$$

$$2{,}365 = \frac{17{,}100{,}000}{(l/r)^2} \qquad \left(\frac{l}{r}\right)^2 = 7{,}230 \qquad \frac{l}{r} = 85$$

$$k = \frac{4{,}730}{2 \times 7{,}230} = 0.327$$

$$\frac{P}{A} = 4{,}730 - 0.327\left(\frac{l}{r}\right)^2 \quad \text{to } \frac{l}{r} = 85 \tag{133.12}$$

$$\frac{P}{A} = \frac{17{,}100{,}000}{(l/r)^2} \quad \text{for } \frac{l}{r} \text{ greater than } 85 \tag{133.13}$$

Equations (133.12) and (133.13) should give the unit stress at failure for these spruce columns under the described conditions of loading. A reasonable working formula may be had by dividing the equations by a factor of safety of 4.

$$\frac{P}{A} = 1{,}200 - 0.082\left(\frac{l}{r}\right)^2 \tag{133.14}$$

Problems

133-9. A spruce column 4- by 6-in. is 6 ft long and is loaded as in Example 4. What safe load would it carry? What load if the length were 10 ft?
Ans. 21,100 lb; 9,500 lb.

133-10. Yellow pine has an ultimate strength of 6,000 psi and a modulus of elasticity of 1,470,000 psi. Derive a hinged-end parabolic formula and its Euler extension on the assumption that the formula will approximate an eccentric ratio of 0.25. Use $\frac{7}{8}L$ for the length in Euler's equation. Apply a factor of safety of 4. *Ans.* $P/A = 1{,}200 - 0.095(l/r)^2$.

133-11. Aluminum alloy 17S-T has a yield strength of 37,000, and the constant, $\pi^2E = 102{,}000{,}000$. Derive a parabolic formula to fit the secant formula with an eccentric ratio of 0.25. Use riveted ends where the length is $\frac{3}{4}L$ and a factor of safety of 3.13. Thence find the total safe load on a 7-in. 6.23-lb aluminum I beam 5 ft long. *Ans.* $P/A = 12{,}000 - 0.612(l/r)^2$.

134. Straight-line Formulas. Some engineers prefer to use a straight-line equation to get the average unit stress as an approximation to the hard-to-solve secant equation. Figure 210 shows one secant curve with an eccentric ratio of 0.25 (which is also shown in Fig. 202) and another curve with an eccentric ratio of 0.1. The effect

[1] *Tech. Paper* 152, U.S. Bureau of Standards, U.S. Government Printing Office 1920.

of eccentricity (which is generally unknown in design) is shown by the area between the secant curves of Fig. 210. If the eccentric ratio ranges from 0.1 to 0.25, as shown in the figure, any curve which lies in the area between these secant curves may agree well with the results of tests. For this reason, straight-line curves have been largely used in practice. A curve of this kind must pass through the ultimate strength for short blocks and approximate Euler's for slender struts. Some straight-line formulas have been derived from experiments and others have been made by drawing a straight line from the ultimate

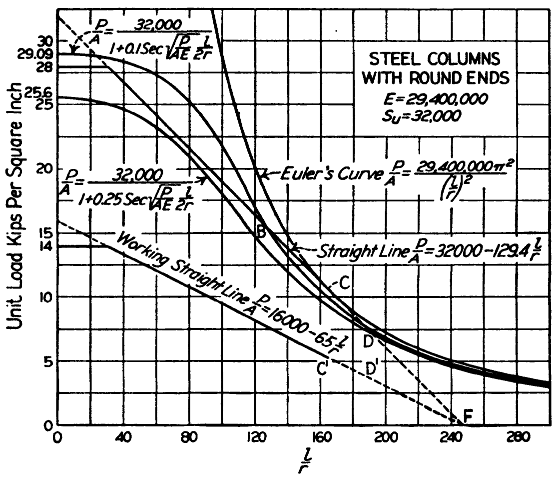


FIG. 210. Straight-line, secant, and Euler's curves.

strength tangent to Euler's. Figure 210 shows a straight line through (0, 32,000) which is tangent to Euler's curve at C. A horizontal stopper, drawn at $P/A = 28{,}000$, replaces the upper part of the straight line.

The straight line crosses the upper secant curve at B and recrosses the lower secant curve at D. Usually the straight line is used to the point of tangency. It might well be extended to D. The lower straight line is a working curve which is obtained by applying a factor of safety of 2. Since the portion CF (or perhaps DF) does not approximate the secant curve, obviously the dotted portion of the working curve, $C'F$ is not valid. For large slenderness ratios, Euler's curve is the best approximation to the secant curve regardless of the eccentric ratio. Therefore, from the point C' to the right (on the graph of Fig. 210), the best curve is Euler's with the same factor of safety as was used for the straight line.

A straight-line equation has the form

$$\frac{P}{A} = S_u - k\frac{l}{r} \tag{134.1}$$

If $P/A = y$ and $l/r = x$, this is recognized as the equation of a straight line with the Y intercept at S_u, and with a negative slope equal to k.

Problems

134-1. Plot Euler's curve for $E = 1{,}440{,}000$ psi. Draw a tangent to this curve from (0, 4,800). Find graphically the abscissa and the ordinate of the point of tangency. Extend the straight line to the X axis. Calculate k. Draw a working curve with a factor of safety of 4, and a stopper of 1,000 psi.

134-2. Plot Euler's curve for $E = 1{,}280{,}000$ psi. Draw the tangent to this curve from (0, 5,400) extending to the X axis. Calculate the slope and write the equation of ultimate strength. Draw a working curve with a factor of safety of 4, and a stopper at 1,200 psi.

134-3. Plot Euler's curve for steel with $E = 29{,}400{,}000$ psi and draw a straight line from (0, 32,000) tangent to it. Divide by a factor of safety to bring the working stress to 18,000 for a slenderness ratio of zero. Draw a horizontal stopper at 15,000. Write the equation of your working formula. This formula is valid for what kind of end conditions?

Ans. $\frac{P}{A} = 18{,}000 - 73\frac{l}{r}$.

While a straight-line formula may always be derived graphically by drawing Euler's curve and plotting the tangent, the methods of calculus are convenient and lead to a simple algebraic result. The problem is that of drawing a straight line tangent to a given curve through a given point which is not on the curve. Euler's formula may be written

$$y = \frac{a}{x^2} \tag{134.2}$$

in which $y = P/A$, $x = l/r$, and $a = \pi^2 E$. It is required to draw a tangent to the curve of Eq. (134.2) which shall pass through the point $(0, S_u)$. The slope of this tangent is

$$\frac{dy}{dx} = -\frac{2a}{x_1^3} \tag{134.3}$$

in which x_1 is the abscissa of the point of tangency. The equation of the tangent line is

$$y = -\frac{2a}{x_1^3}x + S_u \tag{134.4}$$

in which x and y are the coordinates of any point on the line. Since

the point of tangency (x_1, y_1) lies on the straight line of Eq. (134.4), these coordinates satisfy the equation of the line; hence

$$y_1 = -\frac{2a}{x_1^2} + S_u \tag{134.5}$$

Since the point of tangency is on the curve, these coordinates also satisfy Eq. (134.2); hence

$$y_1 = \frac{a}{x_1^2} \tag{134.6}$$

From the above equations, the coordinates of the point of tangency are found to be

$$y_1 = \frac{S_u}{3} \tag{134.7}$$

Equation (134.7) is all that is needed to start the derivation of a straight-line equation. When this ordinate is substituted in Euler's formula, the abscissa of the point of contact is found. The coordinates of the point of tangency and of the Y intercept together determine the equation of the straight line.

Example 1

Derive a straight-line formula for steel which has a yield point of 32,000 and a modulus of elasticity of 29,400,000 psi.

$$\frac{32{,}000}{3} = \frac{\pi^2\, 29{,}400{,}000}{(l/r)^2}$$

$$\left(\frac{l}{r}\right)^2 = 27{,}203 \qquad \frac{l}{r} = 164.93$$

$$k = \frac{2 \times 32{,}000}{3 \times 164.93} = 129.37$$

$$\frac{P}{A} = 32{,}000 - 129.37\,\frac{l}{r} \tag{134.8}$$

gives the ultimate unit load. For a working formula with a factor of safety of 2 for a column with round ends,

$$\frac{P}{A} = 16{,}000 - 65\,\frac{l}{r} \tag{134.9}$$

What formula should be used when l/r is greater than 165? Equations (134.8) and (134.9) are plotted on Fig. 210.

Sometimes two straight-line equations are used instead of one. An example is

$$\frac{P}{A} = 19{,}000 - 100\,\frac{l}{r} \tag{134.10}$$

with a maximum of 13,000 for slenderness ratios not greater than 120, and

$$\frac{P}{A} = 13{,}000 - 50\frac{l}{r} \tag{134.11}$$

for slenderness ratios between 120 and 200.

Problems

134-4. Solve Prob. 134-1 algebraically.

134-5. Solve Prob. 134-2 algebraically.

134-6. Solve Example 1 for a riveted-end column, using $\frac{3}{4}L$ for the length. Apply a factor of safety of 1.78 and a maximum of 15,000 psi.

Ans. $P/A = 18{,}000 - 55\frac{l}{r}$.

134-7. Find the allowable stress on an 8-in. 18.4-lb standard I beam when used as a column 7 ft long by (*a*) Eq. (134.9); (*b*) Formula XXIII; and (*c*) the answer to Prob. 134-6.

134-8. Solve Prob. 134-7 if the length is 14 ft.

134-9. Find the total safe load on the end of a standard 5-in. pipe with flat ends 10 ft long. Could the answer to Prob. 134-6 be used here?

Ans. $P = 219{,}000$ lb.

The Alcoa Structural Handbook gives a table of the ultimate strength formulas for axially loaded aluminum-alloy columns. For two common structural alloys the handbook gives

Alloy	Compressive yield strength	For KL/r less than C	C
17S-T	37,000	$\frac{P}{A} = 43{,}800 - 350\frac{KL}{r}$	83
24S-T	46,000	$\frac{P}{A} = 56{,}600 - 510\frac{KL}{r}$	73

The student will observe that C is merely the slenderness ratio at the point of tangency with Euler's curve. For values greater than C, for all alloys the handbook recommends

$$\frac{P}{A} = \frac{102{,}000{,}000}{(KL/r)^2} \tag{134.12}$$

The factor K takes care of end conditions as in other equations. No reason is given for choosing S_u about 20 per cent higher than the yield strength, and no suggestions are made regarding working formulas.

Problems

134-10. Use a factor of safety of 3.13 on the 17S-T aluminum column formula and write the working equation. What is the real factor of safety based on the yield strength? Find the safe allowable load on a 7-in. 6.23-lb aluminum I beam 5 ft long, using $K = \frac{3}{4}$. *Ans.* $\frac{P}{A} = 14{,}000 - 112\frac{KL}{r}$.

134-11. Using a factor of safety of 3.13 on the 24S-T aluminum column formula, write a working equation. What is the real factor of safety based on the yield strength? Find the total safe allowable load on a 6-in. 8.49-lb aluminum H beam when used as a column with riveted ends ($K = \frac{3}{4}$) if the length is (*a*) 10 ft; (*b*) 14 ft. Use a stopper at 15,000 psi.

Ans. (*a*) 54,400 lb; (*b*) 29,800 lb.

135. Rankine's Formula. Rankine's formula,[1] sometimes called the *Gordon-Rankine formula*, has long been the British favorite, although equations based on the secant formula are now gaining ground. It was the principal formula in America for many years and is still used in some areas of design. It is an *empirical formula*, which gives the unit load equal to the ultimate strength for a short block and approaches Euler's curve for a very long column. The formula is of the form

$$\frac{P}{A} = \frac{S_u}{1 + q(l/r)^2} \tag{135.1}$$

in which S_u is the ultimate unit load in compression on a short block and q is a coefficient, the value of which may be determined experimentally or mathematically from the condition that the curve approaches Euler's for a long column. The allowable unit load is obtained by dividing the numerator by the safety factor, which is the same as taking the allowable compressive stress instead of the ultimate strength as S_u.

The value of q which is derived from the condition that the unit load must approach Euler's value as a limit is called *Ritter's rational constant.* When l/r is zero in Rankine's formula, the denominator is unity and $P/A = S_u$. Rankine's formula, therefore, satisfies one condition. To make it satisfy the other condition, the value of q must be so chosen that the unit load shall be the same in Rankine's and in Euler's formulas for large values of the slenderness ratio.

$$\frac{P}{A} = \frac{\pi^2 E}{(l/r)^2} = \frac{S_u}{1 + q(l/r)^2} \tag{135.2}$$

[1] William J. Rankine (1820–1872), engineer on the Irish railways and later professor at the University of Glasgow, did much to adapt mathematics and precise experiments to engineering design. He investigated the mechanical action of heat, experimented with columns, and revised a formula proposed earlier by Gordon.

For large values of l/r, the second term in the denominator of Rankine's formula is so large relatively that the first term (unity) may be dropped. Then

$$\frac{\pi^2 E}{(l/r)^2} = \frac{S_u}{q(l/r)^2} \tag{135.3}$$

$$q = \frac{S_u}{\pi^2 E} \tag{135.4}$$

This value of q is *Ritter's rational constant.* To make the curve start with the ultimate strength of a short block and approach Euler's curve for long slenderness ratios, does not, unfortunately, assure that the intermediate portion of the curve will fit the results of tested columns. In fact such a curve will not give good values for intermediate lengths of columns, and these columns are the most frequent in design.

Problem

135-1. Find the value of q for steel having a modulus of elasticity of 29,400,000 and an ultimate compressive strength of 32,000 psi. *Ans.* $q = 1/9{,}080$.

Although no one now uses Ritter's constant, the Rankine formula with this constant has been plotted in Fig. 211. Also plotted is another

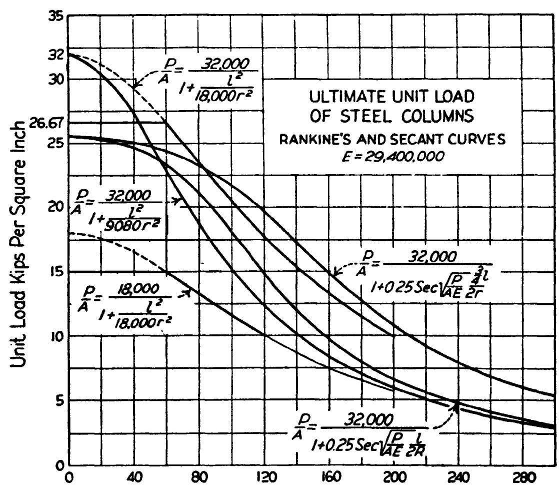


FIG. 211. Comparison of AISC with other curves.

Rankine curve and two secant curves. The approximation is not satisfactory. If a Rankine curve gives the correct unit load for long columns, it is unnecessarily safe for usual columns. If approximately

correct for columns with slenderness ratios below 120, it is unsafe for longer columns.

The bottom curve of Fig. 211 is

$$\frac{P}{A} = \frac{18{,}000}{1 + (l^2/18{,}000r^2)} \tag{135.5}$$

It was much used in the past few years with a maximum (stopper) of 15,000 psi. The portion of this formula between slenderness ratios of 120 and 200 is now recommended for bracing and other secondary members by the AISC. For main compression members with slenderness ratios between 120 and 200, the AISC specifies Eq. (135.5) multiplied by

$$\left(1.6 - \frac{l}{200r}\right) \tag{135.6}$$

This formula is shown in Fig. 209 as the extension of the AISC parabolic formula.

Problems

135-2. Plot the AISC formulas and the ASCE fixed-end formula to a slenderness ratio of 200.

135-3. Calculate the unit load and the total safe load on a 5- by 3¼-in. 11.6-lb Z section when used as a column 6 ft long. Use the AISC, the ASCE formulas, and the answer to Prob. 134-6.

135-4. Solve Prob. 135-3 for a 12-ft length.

135-5. Find the total safe load on a 12- by 12-in. 65-lb wide-flange section as a column 25 ft 2 in. long by AISC formula and Eq. (134.9).
Ans. P = 232,000 lb; 182,000 lb.

135-6. A column with cover plates is made of one wide-flange beam, a 14 by 16 in. 320-lb core section, and two 1- by 18-in. plates which are riveted or welded to the flanges (see handbook). Taking the properties of the beam from the handbook, calculate the moment of inertia and radius of gyration of the section for each principal axis.
Ans. A = 130.12 sq in.; axis perpendicular to web, I = 6,999.5 in.^{4}; r^2 = 53.792; r = 7.334 in.; axis parallel to web, I = 2,607.1 in.^{4}; r^2 = 20.0361; r = 4.476 in.

135-7. Find the total safe load of the column of Prob. 135-6 for a length of 40 ft by the AISC formula and compare with the handbook.

136. Timber Columns. After very thorough tests on timber, the Forest Products Laboratory of the U.S. Department of Agriculture recommended

$$\frac{P}{A} = S\left[1 - \frac{1}{3}\left(\frac{l}{Kd}\right)^4\right] \tag{136.1}$$

in which S is the safe unit stress in compression parallel to the grain for short columns, l is the unsupported length in inches, d is the least transverse dimension

in inches, and K is the value of l/d at the point at which the curve of Eq. (136.1) becomes tangent to Euler's curve with a safety factor of 3. If

$$P/A = y \text{ and } l/d = x,$$

$$y = S - \frac{Sx^4}{3K^4} \tag{136.2}$$

$$\frac{dy}{dx} = -\frac{4Sx^3}{3K^4} \tag{136.3}$$

$$y = \frac{\pi^2 E}{3(l/r)^2} = \frac{\pi^2 E}{36(l/d)^2} = \frac{\pi^2 E}{36x^2} \tag{136.4}$$

$$\frac{dy}{dx} = -\frac{\pi^2 E}{18x^3} \tag{136.5}$$

$$S - \frac{Sx^4}{3K^4} = \frac{\pi^2 E}{36x^2} \qquad 36Sx^2 - \frac{12Sx^6}{K^4} = \pi^2 E \tag{136.6}$$

$$\frac{4Sx^3}{3K^4} = \frac{\pi^2 E}{18x^3} \qquad \frac{12Sx^6}{K^4} = \frac{\pi^2 E}{2} \tag{136.7}$$

From Eqs. (136.6) and (136.7),

$$36Sx^2 = \frac{3\pi^2 E}{2}$$

$$x^2 = \frac{\pi^2 E}{24S} \qquad x' = \frac{\pi}{2}\sqrt{\frac{E}{6S}} \tag{136.8}$$

$$y = \frac{\pi^2 E}{36x^2} \qquad y' = \frac{2S}{3} \tag{136.9}$$

in which x' and y' are the coordinates of the point of tangency. When $y' = 2S/3$ is substituted in Eq. (136.2), $K = x'$.

Problems

136-1. Oak has an allowable compressive stress of 1,000 psi and a modulus of elasticity of 1,600,000 psi. Find the total safe load on an 8- by 10-in. oak post 10 ft long.

136-2. If $S = 1{,}000$ and $E = 1{,}200{,}000$ psi, find the total safe load on a 6- by 6-in. redwood post 10 ft long. *Ans.* $P = 28{,}080$ lb.

136-3. Solve Prob. 136-2 for a length of 15 ft.

$$\textit{Ans. } P = \frac{36 \times 9.87 \times 1{,}200{,}000}{36 \times 30 \times 30} = 13{,}160 \text{ lb.}$$

136-4. If $S = 1{,}200$ and $E = 1{,}200{,}000$ for Douglas fir, find the total safe load on a 6- by 8-in. column 10 ft long.

For solid cylindrical columns of radius d the radius of gyration is $d/4$, while the radius of gyration of a rectangular section is

$$d/\sqrt{12} = 0.2887d.$$

The ratio of the radius of gyration of a circle of diameter d to the radius of gyration of a rectangle of side d is $0.2887 \times 4 = 1.155$. The value of l/d for a cylindrical column must be multiplied by 1.155 to give a slenderness ratio corresponding to that of a rectangular post.

137. Cast-iron Columns. Cast-iron columns are seldom used in structures, although there may be locations where the ability of cast iron to resist corrosion may make it desirable. Modern methods of centrifugal casting produce pipe of uniform thickness and quality which may be used with confidence. One established equation for cast iron is

$$\frac{P}{A} = 9{,}000 - 40\frac{l}{r} \tag{137.1}$$

with a maximum slenderness ratio of 70. This is conservative and is recommended as a working formula.

Problems

137-1. A cast-iron column is 11 in. outside diameter and 10 in. inside. Find the total safe load for a 16-ft length. *Ans.* $P = 114{,}000$ lb.

137-2. Derive a straight-line round-end column formula for cast iron having an ultimate strength of 45,000 and a modulus of elasticity of 15,000,000 psi, using a factor of safety of $4\frac{1}{2}$.

137-3. By the formula of the preceding problem, find the total safe load on a hollow cast-iron pipe of 6 in. outside and 4 in. inside diameter for lengths of 10 and 20 ft. *Ans.* $P = 87{,}000$ lb; 29,000 lb.

138. Selection of Column for a Given Load. The problem of designing or selecting a column of a given length to carry a given load varies with the form of the section. If the sections which are considered are all similar figures, the radius of gyration varies as the first power and the area varies as the second power of any dimension. For a circle of radius a, for instance, $r = a/2$ and $A = \pi a^2$. For a square of side b, $r = b/\sqrt{12}$ and $A = b^2$. A problem of this class may be solved algebraically for the unknown dimension. Euler's equation gives the fourth power of this unknown quantity (since the moment of inertia varies as the fourth power). The required result is obtained by extracting the square root of a square root. A straight-line formula gives a quadratic equation. Rankine's formula gives a quadratic equation in terms of the square of the unknown dimension. Any one of these equations may be easily solved.

Example 1

A square steel bar, as a column 15 ft long, carries a load of 15,000 lb. Find its dimensions as a round-end column with a factor of safety of 3, using $E = 29{,}400{,}000$.

If the factor of safety is applied to the total load, the calculation is made for a total load of 45,000 lb. It is evident that the required area will not be greater than 10 sq in., which makes the radius of gyration small enough to use Euler's formula.

$$\frac{45{,}000}{b^2} = \frac{\dfrac{\pi^2 \times 29{,}400{,}000}{180 \times 180 \times 12}}{b^2}$$

$$b^4 = \frac{45{,}000 \times 180 \times 180 \times 12}{\pi^2 \times 29{,}400{,}000} = \frac{18^2 \times 90}{\pi^2 \times 49}$$

$$b^2 = \frac{54\sqrt{10}}{7\pi} = 7.765 \text{ in.}^2$$

$$b = 2.787 \qquad \frac{P}{A} = \frac{15{,}000}{7.765} = 1{,}932 \text{ psi}$$

Example 2

Solve Example 1 for a solid circular rod of radius r.

$$r^4 = 6.3976 \text{ in.}^4 \qquad r^2 = 2.5293 \text{ in.}^2 \qquad r = 1.5904 \text{ in.}$$

$$\frac{P}{A} = 1{,}889 \text{ psi} \qquad \text{slenderness ratio} = 226.3$$

Example 3

By Eq. (134.9), find the diameter of a steel cylindrical column 5 ft long to carry 60,000 lb.

$$\frac{P}{A} = 16{,}000 - 65\frac{l}{r}$$

$$16{,}000 - \frac{65 \times 60 \times 4}{d} = \frac{60{,}000 \times 4}{\pi d^2}$$

$$d = 2.73 \text{ in.} \qquad \text{slenderness ratio} = 87.9 \qquad \frac{P}{A} = 10{,}260 \text{ psi}$$

Problems

138-1. Solve Example 3 for a load of 80,000 lb.

138-2. Solve Example 3 for a hollow cylinder with outside diameter twice the inside diameter.

138-3. Solve Example 3 by the ASCE hinged-end formula.

138-4. Find the size of a square timber column 10 ft long to carry 24,000 lb by the formula $\dfrac{P}{A} = 1{,}200 - 10\dfrac{l}{r}$.

138-5. A rectangular timber column is 10- by D-in. by 8 ft long. It is to carry 60,000 lb by the formula of Prob. 138-4. Find the dimension D.

138-6. Find the size of a square aluminum column 6 ft long to carry 120,000 lb by the formula of Prob. 133-11.

Since the sections of rolled shapes of different sizes are not similar figures, the selection of a column must be made by trial and error. The Steel Construction Manual gives the strength of steel columns and some fabricated sections, which have been calculated by the AISC formula. When another formula is required by the specifications, the approximate size may be selected from the table and the remaining calculations completed by trial. An example will make this clear.

Example 4

Select a wide-flange section for a column 20 ft long to carry 240,000 lb, by Eq. (134.9).

A 12- by 12-in. wide-flange 65-lb section will carry 266 kips, according to the AISC tables. Its least radius of gyration is 3.02 in., which, substituted in the straight-line equation, gives

$$\frac{P}{A} = 10{,}840 \text{ psi} \qquad P = 19.11 \times 10{,}840 = 208{,}000 \text{ lb}$$

If r can be held to this value approximately, then we shall need a column with an area $= 240{,}000/10{,}840 = 22.10$ sq in. A 12- by 12-in. wide-flange 79-lb section has $A = 23.22$ and $r = 3.05$ and will be satisfactory.

Problems

138-7. In Example 4, instead of a 12- by 12-in. section, select a 10- by 10-in. wide-flange and a 14- by 12-in. wide-flange section which will carry the load.

138-8. Select a standard steel I beam to be used as a column 8 ft. long and to carry 25,000 lb by the ASCE fixed-end formula.

138-9. Select a standard steel I beam to be used as a column 12 ft long and to carry 130,000 lb by the ASCE fixed-end formula.

138-10. Select a wide-flange section to be used as a column 30 ft long and to carry 400,000 lb by the formula of Prob. 134-6.

138-11. Select a standard steel pipe 16 ft long to carry 100,000 lb by the ASCE formula for fixed ends.

139. Web Crippling of Beams. Certain parts of beams may fail as columns. Such failure is due to stability rather than any lack of strength in the material. If a deep I beam has a thin web, the shearing stresses in the web may cause it to buckle. An approximate solution is obtained by considering the average shearing stress in the web from Formula XVI, Art. 72.

$$S_s = \frac{V}{td} \qquad \text{Formula XVI}$$

in which V is the total vertical shear, t is the thickness of the web, and d is the total depth. The product $t \times d$ is sometimes designated by A, meaning the area of the web regarded as extending the entire depth of the beam.

Beams on short spans may carry heavy loads without exceeding the allowable bending stress, but the shearing stress may be excessive. The student may think that the stress given by Formula XVI is vertical because V is vertical. But it must be remembered that there is an equal shearing stress horizontally, as was shown in the element of

Fig. 30,I. If we think of this element as located midway between F and G in Fig. 212, we observe that there is a resultant compressive stress at 45° or in the direction of FG. (At the ends of the strip FG there is also a tensile or compressive stress which alters the direction of the maximum resultant compressive stress, but it is customary to use the 45° strip, probably because the resultant compressive stress is equal to the applied shearing stress (see Formula VI, Art. 22). If the entire web of the I beam is regarded as made up of a series of parallel columns with fixed ends, such as FG of Fig. 212, each column may be assumed to be 1 inch wide. Its thickness is t, the thickness of the web.

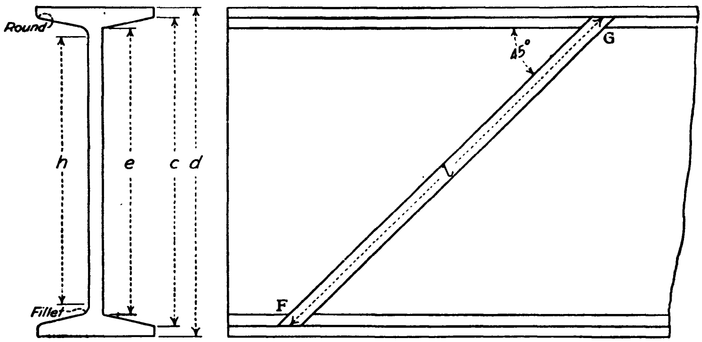


FIG. 212. I-beam web as a diagonal column.

If h is the clear distance between the flanges, the length of this column is $\sqrt{2}h$. Since the radius of gyration of a rectangular section is $t/\sqrt{12}$,

$$\text{Slenderness ratio} = \frac{\sqrt{2}h}{t/\sqrt{12}} = \frac{2h\sqrt{6}}{t} \tag{139.1}$$

It is customary to regard the web as perfectly fixed at the ends with a load of very little eccentricity. Under these conditions, the free length is one-half of l and the equivalent slenderness ratio is $\frac{h\sqrt{6}}{t}$. Euler's formula for fixed ends then becomes

$$\frac{P}{A} = s_c = s_s = \frac{\pi^2 E}{6(h/t)^2} = \frac{1.64E}{(h/t)^2} \tag{139.2}$$ [1]

[1] *Bulletin* 86, University of Illinois Engineering Experiment Station, Strength of Webs in I beams and Girders, by Profs. H. F. Moore and W. M. Wilson, is an extensive theoretical and experimental study. This equation is in the form given in this bulletin.

which gives the ultimate strength of the web in resistance to diagonal buckling.

Many formulas for diagonal buckling have been given in handbooks and specifications, which differ greatly as to the allowance which should be made for this possibility of failure. The AISC (1947) specifies 13,000 psi as the allowable shearing stress in the webs of beams and plate girders for values of h/t up to 70. This agrees with the results of a series of tests at Lehigh University[1] where there was no evidence of buckling with ratios up to 70. Beyond this ratio, the AISC requires stiffeners where the shearing stress exceeds $\frac{64{,}000{,}000}{(h/t)^2}$. Since no standard rolled I beam or wide-flange section has a ratio of h/t equal to 70, the spacing of stiffeners needs investigation only on built plate girders.

This is called *web buckling* and is caused by lateral instability of the web as a unit. Local failure produced by excessive compressive stresses under a concentrated load or reaction is called *web crippling*. The allowable compressive stress on webs of rolled-steel sections at the toe of the fillet is 24,000 psi, according to AISC specifications. For concentrated loads

$$\frac{R}{t(N + 2k)} = \text{not over } 24{,}000 \text{ psi}$$

where R = concentrated load or reaction
t = thickness of web, in inches
N = length of bearing under load, in inches
k = distance from outer face of flange to web toe of fillet

For end reactions, substitute k for $2k$.

Example 1

Find the maximum reaction permissible at the support for a 16- by 7-in. 40-lb wide-flange beam if the width of the support is 3½ in. Find the width of support necessary to take the full web shear.

$$13{,}000 = S_s = \frac{V}{td} = \frac{R}{16 \times 0.307}$$

$$R = 63{,}860 \text{ lb.} \quad \text{(Check with AISC Manual.)}$$

$$24{,}000 = \frac{R}{0.307(3\frac{1}{2} + 1)} \qquad R = 33{,}160 \text{ lb}$$

[1] LYSE, INGE, and H. J. GODFREY, Web Buckling in Steel Beams, *Trans. ASCE*, Vol. 61, No. 100, p. 675, 1935.

This is the maximum reaction at the end with a 3½-in. bearing.

$$24{,}000 = \frac{63{,}860}{0.307(N + 1)} \qquad N = 7.66 \text{ in.}$$

This is the required length of end support bearing if the web is stressed to 13,000 psi.

The maximum concentrated load in the middle of the span would be 127,720 lb. The length of bearing under the load (on the top flange) to keep web-crippling stresses within allowable values may be found.

$$24{,}000 = \frac{127{,}720}{0.307(N + 2)} \qquad N = 15.32 \text{ in.}$$

Problems

139-1. Solve the example above for a 20-in. 95-lb standard I beam and check with the AISC manual.

139-2. Solve the example for a 12- by 8-in. 50-lb wide-flange beam. Check with handbook.

Example 2

A built-up steel plate girder is made of one 42- by ¼-in. web plate, four 6- by 8- by 1-in. angles (with the 8-in. legs horizontal), and two 1- by 17-in. flange plates. Using Eq. (139.2) with a factor of safety of 1.5, find the maximum allowable shear if used without stiffeners.

$$h = 30 \text{ in.} \qquad t = \tfrac{1}{4} \text{ in.} \qquad \frac{h}{t} = 120$$

$$\frac{P}{A} = S_c = S_s = \frac{1.64 \times 30{,}000{,}000}{1.5 \times 144{,}000} = 2{,}291 \text{ psi}$$

$$V = 2{,}291 \times 44 \times \tfrac{1}{4} = 25{,}100 \text{ lb}$$

Problem

139-3. A plate-and-angle girder is made of a 39½- by ⅜-in. web plate and four 5- by 3½- by ⅜-in. angles with the short legs connected to the web plate. The total depth of the girder is 40 in. Find the allowable unit shearing stress if no stiffeners are used.

140. Lateral Buckling of the Compression Flange. The compression flange of a beam may fail as a column by lateral deflection. For 50 years engineers have struggled with the problem of elastic stability of the I beam. The highly complex and theoretical solutions also involve the torsional rigidity of the beams and are too difficult for the engineer to apply in designing. Former formulas based on the ratio of the unsupported length to the flange width are inadequate as a criterion for safety and accuracy.

A simple experiment with a yardstick supported on edge at its ends and loaded at the middle will illustrate that stability, rather than strength, will be the criterion of the amount of load than can be placed on it. The stick will deflect laterally even if the load is hung by a

string. A long I beam with narrow flanges will behave the same way. All specifications for beam design provide for reducing the allowable bending stress in the compression flange as the unsupported span length is increased. In a recent study[1] the critical stress, computed by an extension of Prof. Timoshenko's method,[2] was plotted against the ratio ld/bt (where l is the unsupported length of the compression flange, d is the depth of the beam, b is the width of the compression flange, and t is the thickness of the compression flange). The plotted points were so distributed that it was possible to write equations for the curves. When a factor of safety was introduced, the stress in a carbon-steel beam could be expressed

$$S = 18{,}000 - 0.006\left(\frac{ld}{bt}\right)^2 \tag{140.1}$$

For values of ld/bt over 1,000, the following formula should be used

$$S = \frac{12{,}000{,}000}{ld/bt} \tag{140.2}$$

The AISC Manual of Steel Construction has computed the value of d/bt for all I beams and wide-flange sections. Their specifications, however, recommend the use of a 20,000-psi allowable bending stress for all values of ld/bt up to 600, and the use of Eq. (140.2) for higher values.

Problems

140-1. By Eq. (140.1), find the allowable bending stress for a 14- by 6¾-in. 30-lb wide-flange beam supported at the ends on a 10-ft span. Then find the total uniformly distributed load that may be placed on the beam.
Ans. $S = 15{,}500$ psi; $W = 28{,}800$ lb.

140-2. How much total uniform load can be placed on the beam of Prob. 140-1 by the AISC specifications? *Ans.* 37,100 lb.

140-3. Solve Probs. 140-1 and 140-2 for a span of 16 ft. *Ans.* $W = 21{,}600$ lb.

140-4. Find the total uniform load that may be placed on a 16- by 7-in. 36-lb wide-flange beam when supported at the ends on an 8-ft span. Use both Eq. (140.1) and the AISC specifications.

140-5. Solve Prob. 140-4 for a length of 15 ft.

140-6. For the 16- by 7-in. 36-lb wide-flange beam on a 15-ft span, how many lateral braces would be needed to support the compression flange if the bending stress is 20,000 psi?
Ans. One at the middle will keep $ld/bt < 600$.

[1] DE VRIES, KARL, Strength of Beams as Determined by Lateral Buckling, *Trans. ASCE*, Vol. 112, p. 1245, 1947.

[2] TIMOSHENKO, S., "Theory of Elastic Stability," McGraw-Hill Book Company, Inc., New York, 1936, Chap. V.

141. Column Failure by Flange Buckling at Edge. A column as a whole may be sufficiently rigid to carry the required load but may begin to fail by lateral buckling of the edge of a thin flange. Figure 213,I shows a fabricated T section which was used by the ASCE committee to study this kind of failure.[1]

A $\frac{3}{8}$-inch plate is riveted to two 4- by 4-inch angles. For most of the experiments these angles were $\frac{3}{8}$ inch. Since these bent, the free width w of the plate (which will hereafter be called the *flange*) is taken from the center of the rivet holes. When $\frac{3}{4}$-inch angles were used, they were found to be so rigid that the free width of the flange was measured from the toe to the angle. The free outstanding widths varied from 4.29 to 11.17 inches. The length of all test pieces was 75 inches.

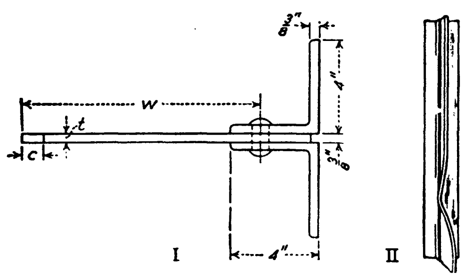


FIG. 213. Flange buckling.

If a small strip of width c is regarded as cut off from the edge of the flange, this strip would form a column with a slenderness ratio of 693, which would carry a very small load. The deflection of the strip is restrained by the remainder of the plate, which acts as a horizontal cantilever fixed at the angles. If the flange is too wide, the load carried by this strip at the edge fails to offset the damage done by its bending moment, and the ultimate load on the section is not increased by the increased area. If w/t is the ratio of the outstanding width to the thickness of the flange, these tests showed that no increase in total load was secured when this ratio was increased from 15 to 20. From these tests, it is evident that a free flange width greater than fifteen times the thickness represents a waste of material.

Figure 213,II shows how a flange may buckle under load as a column. The free edges of the flanges of a wide-flange beam or H beam may buckle in a similar way.

[1] Final Report of the Special Committee on Steel Column Research, *Trans. ASCE*, Vol. 59, No. 98, pp. 1435–1444, 1933.

Great advances were made during the war in forming structural shapes from sheet metal which were manufactured in various shapes by automatic spot welding. This led to the study of thin compression flanges when used in columns and in beams. The results[1] of beams indicate that compression flanges stiffened along both longitudinal edges and having a ratio of b/t up to 25 were fully effective and failed by simple yielding. (In this study b is the width of the outstanding leg of the compression flange and t is its thickness). For flanges stiffened along one longitudinal edge (the web), failure occurred by simple yielding up to ratios of 12.

Specifications of the AISC permit compression flanges of double angles, flanges of I beams, and flanges or stems of tees to have width-thickness ratios of 16.

[1] WINTER, GEORGE, Strength of Thin Steel Compression Flanges, *Trans. ASCE*, Vol. 112, p. 527, 1947.

CHAPTER 15

COMBINED STRESS

142. Resultant of Shearing and Tensile Stress. In the simply supported beam of Fig. 214,I there will be a horizontal tensile stress on the element at A. This is computed by the usual flexure formula $S = Mv/I$. There will also be a shearing stress $S_s = VQ/Ib$, which will be upward on the left side of the element, as can be seen from the shear diagram. The forces on the element A are shown in Fig. 214,II.

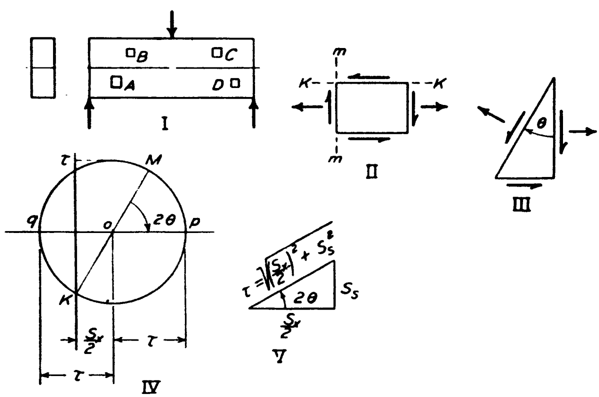


FIG. 214.

Let the area tributary to the tensile stress be A and cut the element by an inclined plane as shown in Fig. 214,III. By resolutions,

$$S_s'A \sec \theta = S_xA \sin \theta - S_sA \cos \theta + S_sA \tan \theta \sin \theta \quad (142.1)$$

$$S_s' = \tfrac{1}{2}S_x \sin 2\theta - S_s \cos 2\theta \quad (142.2)$$

where S_s' is the shearing stress on the inclined plane making an angle θ with the area A.

Similarly,

$$S_t'A \sec \theta = S_xA \cos \theta - S_sA \sin \theta + S_sA \tan \theta \cos \theta \quad (142.3)$$

$$S_t' = \tfrac{1}{2}S_x(1 + \cos 2\theta) + S_s \sin 2\theta \quad (142.4)$$

Differentiating Eq. (142.2),

$$\tan 2\theta = -\frac{\tfrac{1}{2}S_x}{S_s} \tag{142.5}$$

And substituting back,

$$\text{Maximum shearing stress} = \tau = \sqrt{(\tfrac{1}{2}S_x)^2 + S_s^2} \tag{142.6}$$

By similar methods on Eq. (142.4),

$$\tan 2\theta = \frac{2S_s}{S_x} \tag{142.7}$$

$$\text{Maximum normal stress} = p = \tfrac{1}{2}S_x + \tau \tag{142.8}$$

$$\text{Minimum normal stress} = q = \tfrac{1}{2}S_x - \tau \tag{142.9}$$

The above equations may be solved by Mohr's circle. The tensile stress on the left face *mm* in Fig. 214,II (positive normal stress), and the shearing stress (upward on the left face, or clockwise couple for the vertical shears, is positive) are plotted at M in Fig. 214,IV. The shearing stress on the top face *kk* is negative (counterclockwise couple formed by horizontal shears) and is plotted at K, since there is no normal stress on *kk*. The radius of the circle which is drawn with MK as diameter gives the maximum shearing stress, and the intersections at p and q give the maximum and minimum normal stresses. In Fig. 214,IV, the maximum tensile stress p is located at an angle 2θ clockwise from M. Therefore, the plane on which the maximum tensile stress occurs will be located θ clockwise from the vertical face as shown in Fig. 214,III. The stress on any other plane may be found in a similar manner. The proof of this will be given in the next article.

Example 1

At the point A in the beam of Fig. 214, the bending stress is 12,000 psi and the shearing stress 8,000 psi. Find the maximum resultant stresses at A and draw free bodies to show their planes.

Point M with coordinates (12,000, 8,000) is located. Point K with coordinates (0, −8,000) is located. The planes of these applied stresses are 90° apart, hence M and K are 180° apart and MK is a diameter of the circle. The intercepts give p = 16,000 psi tension and q = 4,000 psi compression. The radius of the circle is τ = 10,000 psi. To find the plane of p,

$$\tan 2\theta_m = {}^{8000}\!/_{6000}$$
$$2\theta_m = 53°08' \text{ clockwise}$$
$$\theta_m = 26°34' \text{ clockwise}$$

The angle is shown on Fig. 215,II, the free body containing the plane of maximum tensile stress. Figure 215,III shows the free body with the plane of maximum compression. In IV and V are shown the planes of maximum shear. The direc-

tion of the shear on the left inclined face is upward for positive and downward for negative shear, as obtained from the ordinate of the circle.

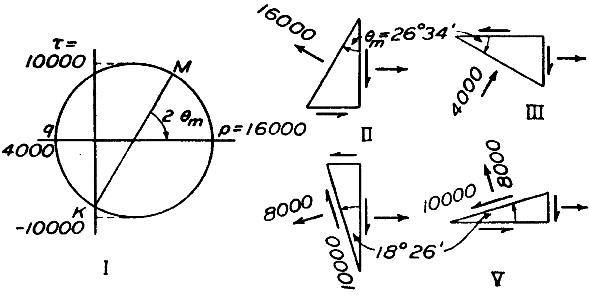


FIG. 215.

Example 2

Find the stresses at the point A in Fig. 214,I, on a plane making an angle of 45° counterclockwise from the vertical line through the element.

Some point on the circle of Fig. 215,I will have coordinates which give the required stresses. The point will be 90° counterclockwise from point M, since the coordinates of M are the stresses on the vertical edges of the element A. The radius vector to the desired point will make an angle of 36°52′ with a horizontal line to q. The stresses are $S_c' = 2{,}000$ psi and $S_s' = 6{,}000$ psi.

Example 3

A 6- by 10-in. beam is supported at points 30 in. apart and carries a load of 20,000 lb midway between the supports. Find the magnitude and direction of the maximum resultant tension, shear, and compression, at sections 5 and 10 in. from the left support at points 0, 1, 2, 3, 4, and 5 in. from the neutral axis.

Table 18 gives the results of the calculation for this problem. It will be noticed that the tension is at 45° with the horizontal at the neutral surface and is 250 psi. At 5 in. from the end the resultant tensile stress increases to 500 psi in the outer fibers, and at 10 in. from the end it increases to 1,000 psi.

It is not usually necessary to calculate the maximum resultant tensile stress in a beam, since it is seldom greater than the bending stress in the outer fibers. Since the shear is a maximum at the neutral surface, where the bending stress is zero, and the tension is a maximum in the outer fibers, where the shear is zero, the maximum resultant stress at any point in a beam is seldom greater than the stress in the outer fibers which is due to bending alone. In a short I beam, the resultant tensile or compressive stress in the web may be greater than the stress in the outer fibers. For the 24- by 12-inch 100-pound wide-flange section of Prob. 72-3, the unit shearing stress at 10.5 inches from the neutral surface was found to be $0.0800V$. If $V = 120{,}000$ pounds,

TABLE 18. RESULTANT SHEAR AND TENSION IN A BEAM

Distance below axis		Shear, psi	Tension, psi	Maximum shear		Maximum tension		Maximum compression	
				Psi	Angle	Psi	Angle	Psi	Angle
At 5 in. from end	0	250	0	250.0	0° 0′	250.0	−45° 0′	250.0	45° 0′
	1	240	100	245.2	5°53′	295.2	−39°07′	195.2	50°53′
	2	210	200	232.6	12°44′	332.6	−32°16′	132.6	57°44′
	3	160	300	219.3	21°35′	369.3	−23°25′	69.3	66°35′
	4	90	400	219.0	32°53′	419.0	−12°07′	19.0	77°53′
	5	0	500	250.0	45° 0′	500.0	0° 0′	0	90° 0′
At 10 in. from end	0	250	0	250.0	0° 0′	250.0	−45° 0′	250.0	45° 0′
	1	240	200	260.2	11°49′	360.2	−33°11′	160.2	56°49′
	2	210	400	290.0	21°48′	490.0	−23°12′	90.0	66°48′
	3	160	600	340.0	30°58′	640.0	−14° 2′	41.0	75°58′
	4	90	800	410.0	38°40′	810.0	− 6°20′	10.0	83°40′
	5	0	1,000	500.0	45° 0′	1,000.0	0° 0′	0	90° 0′

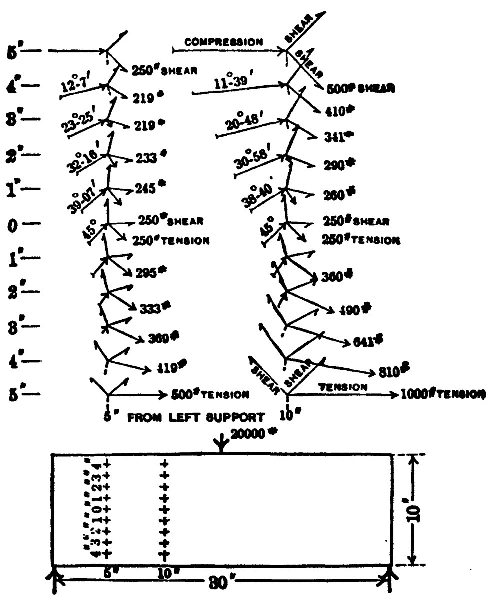


FIG. 216. Resultant stress in a beam section.

$s_s = 9{,}600$ pounds per square inch. If the bending stress in the outer fibers is 16,000 pounds per square inch, the tensile stress 10.5 inches below the neutral axis is 14,000 pounds per square inch.

$$p = 7{,}000 + 11{,}880 = 18{,}880 \text{ psi}$$

For short, deep beams it is desirable to investigate for maximum resultant tension or compression in the web.

In a reinforced-concrete beam, the steel in tension is mathematically equivalent to a very wide flange of concrete. The unit shearing stress in the concrete which adjoins the reinforcement is large. Since the steel is near the outer surface, the bending stress in the concrete is also large. The resultant tensile stress in the concrete is relatively large, and such beams often begin to crack along surfaces at right angles to the direction of the maximum resultant tension.

Problems

142-1. At a point D in the beam of Fig. 214,I, the bending stress is 800 psi and the shearing stress is 300 psi. Find the maximum resultant stresses and draw free bodies to show the planes where these stresses occur.

142-2. Find the stresses in Prob. 142-1 at point D on a plane θ making an angle of 75° as shown in Fig. 214,III.

142-3. At the point C in Fig. 214,I, the bending stress is 2,400 psi and the shearing stress is 500 psi. Find the resultant stresses and show on free-body sketches the planes of these stresses.

142-4. Find the stresses in Prob. 142-3 at point C on a plane θ making an angle of 60° as shown in Fig. 214,III.

142-5. A 4- by 8-in. wood beam 6 ft long is supported at the ends and carries a 3,200-lb load concentrated at the middle of the span. Find the maximum bending stress and the maximum shearing stress and show on your sketch exactly where they occur. At a point B, 2 in. below the top surface of the beam and 2 ft from the left support, find the applied shearing and normal stresses on the element. Then find the maximum resultant stresses at that point and show their planes on sketches.

Ans. $S = 1{,}350$ psi; $S_s = 75$ psi; $\tau = 232$ psi; $p = 7$ psi; $q = -457$ psi.

142-6. In Prob. 142-5 at the point B, find the planes where the resultant normal stress is zero. What is the shearing stress here?

142-7. A 14-in. by 6¾-in. 34-lb wide-flange section is 6 ft long and supported at the ends. It carries a uniform load of 20,000 lb per ft distributed over the left half of the span and no load on the other half. Neglect the beam's own weight and find (a) the maximum bending stress; (b) the bending stress at point A, 4 ft from the left end and just under the top flange; (c) the shearing stress at A, by the exact formula and check by the approximate (and usual I beam) formula; (d) the resultant normal and shearing stresses at A; and (e) the planes where these stresses occur.

Ans. $S = 13{,}900$ psi; $S_s = 3{,}190$ psi; S_s approx $= 3{,}780$ psi; $\tau = 7{,}650$ psi; $p = 14{,}600$ psi; $q = -700$ psi.

143. Proof of Mohr's Circle. That the construction of the circle gives correct results for p, q, and τ is evident from Eqs. (142.6) to (142.9) and Fig. 214. To show that the stresses on any plane nn of Fig. 217,III may be obtained from the circle, proceed as follows:

Let plane pp of Fig. 217,II be the plane θ_m of maximum tensile stress p. This plane is defined in Fig. 214,V. Let nn be any other plane θ.

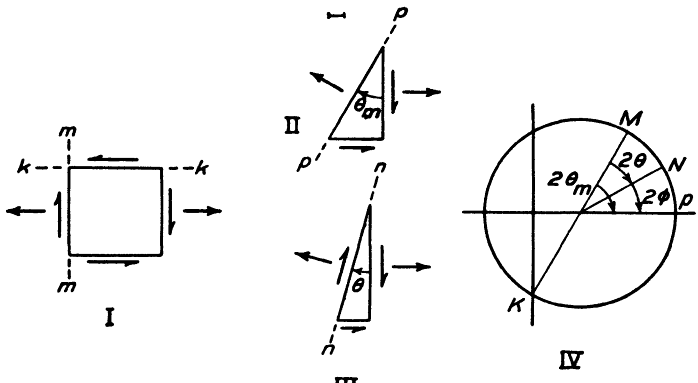


FIG. 217.

The stresses on mm and kk of Fig. 217,I are plotted in IV and the circle drawn, giving p and θ_m. In Eq. (142.4), substitute $\theta_m - \phi$ for θ.

$$S_t' = \tfrac{1}{2}S_x(1 + \cos 2\theta_m \cos 2\phi + \sin 2\theta_m \sin 2\phi) + S_s(\sin 2\theta_m \cos 2\phi - \cos 2\theta_m \sin 2\phi) \quad (143.1)$$

The trigonometric values for $2\theta_m$ are substituted from the triangle of Fig. 214,V.

$$S_t' = \frac{1}{2} S_x \left(1 + \frac{S_x}{2\tau}\cos 2\phi + \frac{S_s}{\tau}\sin 2\phi\right) + S_s\left(\frac{S_s}{\tau}\cos 2\phi - \frac{S_x}{2\tau}\sin 2\phi\right) \quad (143.2)$$

$$S_t' = \tfrac{1}{2}S_x + \tau \cos 2\phi \quad (143.3)$$

which is the abscissa of N.

Problems

143-1. Using the method above, show that the ordinate of N will check the circle.

143-2. Using the method above with an angle ϕ larger than θ_m, show that the equations will check the coordinates of N.

144. Bending Combined with Torsion. In a shaft subjected to bending moment, the maximum tensile stress is found at the dangerous

section in the fibers which are most remote from the neutral surface. When subjected to torsion, all the outer fibers are at the maximum shearing stress. When the shaft is subjected to the combined effect of bending moment and torque, those fibers at the dangerous section which are farthest from the neutral surface are subjected to the combined effect of the maximum tensile or compressive stress and the maximum shearing stress, which may be much larger than the results of Formulas IX and XIII.

Example

A 1-in. rod projects from a vise. A wrench, at right angles to the rod, grips it 8 in. from the vise. The wrench is turned by a force of 60 lb, perpendicular to the plane of the rod and wrench, which is applied to the wrench 12 in. from the axis of the rod. Find the maximum resultant shearing and tensile stress.

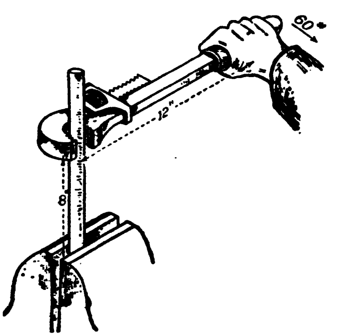


FIG. 218. Torsion and bending.

The bending moment at the vise is the same as if the force of 60 lb were applied directly to the rod at 8 in. from the vise (Fig. 218).

$$M = 60 \times 8 = 480 \text{ in.-lb}$$

$$S_t = \frac{480 \times 32}{\pi} = \frac{3{,}840 \times 4}{\pi} = 4{,}889 \text{ psi}$$

$$T = 60 \times 12 = 720 \text{ in.-lb}$$

$$S_s = \frac{720 \times 16}{\pi} = \frac{3{,}840 \times 3}{\pi} = 3{,}667 \text{ psi}$$

$$\tau = \sqrt{3{,}667^2 + 2{,}444^2} = 4{,}407 \text{ psi}$$

$$p = 2{,}444 + 4{,}407 = 6{,}851 \text{ psi}$$

Since the section modulus used in torsion is twice that used in bending and the force P is the same for both torque and bending moment, there is a large common factor which may be taken out to reduce the labor of computation. In this problem the factor is $3{,}840/\pi$, which is equal to 1,222.

$$\tau = 1{,}222 \sqrt{3^2 + 2^2} = 1{,}222 \sqrt{13} = 4{,}407 \text{ psi}$$

Problems

144-1. A solid 3-in. shaft, which projects from a vise, is twisted by a pipe wrench applied 40 in. from the vise. A force of 600 lb at right angles to the plane of the shaft and the wrench is applied to the wrench 60 in. from the axis of the shaft. Find the maximum resultant shearing and tensile stress.

Ans. $\tau = 8{,}161$ psi; $p = 12{,}688$ psi.

144-2. A 3½-in. standard pipe projects from a vise and is twisted by a force of 400 lb applied 5 ft from the axis of the pipe in a plane which is 4 ft from the vise and perpendicular to the axis of the pipe. Find the maximum resultant shearing and tensile stress. *Ans.* $\tau = 6{,}419$ psi; $p = 10{,}429$ psi.

144-3. A 2-in. diameter rod projects from a vise as shown in Fig. 218. The wrench grips the rod at 20 in. from the vise and the 314-lb force is applied 15 in. from the center of the rod. Find the maximum resultant stresses at a

point A which is just above the vise and on the front of the rod. Draw free bodies to show the planes of the stresses.

144-4. Solve Prob. 144-3 for a point B just above the vise and on the rear of the rod. *Ans.* $p = 9{,}000$ psi at 18°26′ with the horizontal and clockwise; $q = -1{,}000$; $\tau = 5{,}000$ psi.

144-5. A solid 4-in. shaft is subjected to a compressive force of 80,000 lb longitudinally and a torque of 6,000 ft-lb. Find the maximum resultant shearing and compressive stress.

144-6. A 4-in. solid shaft transmits 200 hp at 150 rpm and is subjected to a compression of 40,000 lb in the direction of its length. Find the maximum resultant compressive and shearing stresses.

Ans. $\tau = 6{,}874$ psi; $q = 8{,}465$ psi.

144-7. A 4-in. diameter shaft transmits 98.7 hp at 330 rpm and is subjected to direct compression endwise of 31,400 lb. The driving motor is on the right end of the shaft and rotates the shaft clockwise, as viewed on the right end. Find the maximum resultant stresses and the planes where they occur. Show the planes on free bodies.

144-8. A nominal 4-in. double-extra-strong pipe column section is clamped vertically in a vise and twisted by a force of 600 lb at the end of a 6-ft pipe wrench (measured to the center of the pipe). The wrench is horizontal and 5 ft from the vise, as in Fig. 218. The outside diameter of the pipe is 4,500 in; the inside diameter is 3.152 in. (See handbook for other details). Find the maximum resultant stresses occurring at point A, just above the vise and on the front of the pipe. Draw sketches showing planes of these stresses.

145. Shear Combined with Tension in Two Directions. In Fig. 219,I, a unit cube, subjected to biaxial tension and shear in the same

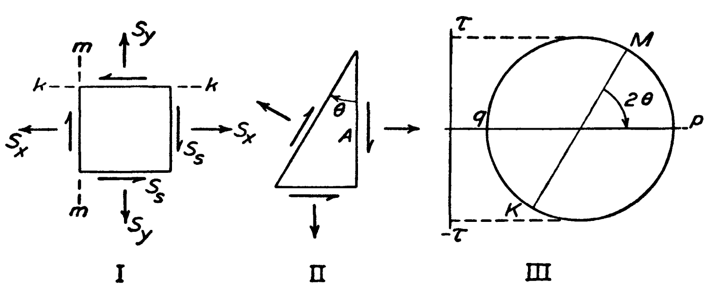


FIG. 219.

plane, is shown. Cutting the cube and writing resolution equations, as was done in Art. 142, will give expressions for the normal and shearing stresses on the inclined plane of Fig. 219,II. Following the method of differentiating and substituting, the maximum stresses are found:

$$\tau = \sqrt{S_s^2 + \left(\frac{S_x - S_y}{2}\right)^2} \tag{145.1}$$

$$p = \frac{S_x + S_y}{2} + \sqrt{S_s^2 + \left(\frac{S_x - S_y}{2}\right)^2} \tag{145.2}$$

If the stress is compressive, it may be regarded as negative tension.

Problems

145-1. A block is subjected to a horizontal tensile stress of 1,000 psi, a vertical tensile stress of 200 psi, and a horizontal and vertical shearing stress of 300 psi which is upward at the left. Find the maximum unit shearing and tensile stress. *Ans.* $\tau = 500$ psi; $p = 1{,}100$ psi.

145-2. Solve Prob. 145-1 if the stress of 200 psi is compressive.

145-3. Derive Eqs. (145.1) and (145.2) together with the equation for minimum normal stress. Find the angles where these maximum stresses exist.

145-4. Show that the equations of Prob. 145-3 may be solved by Mohr's circle.

145-5. An element is subjected to a horizontal tensile stress of 1,500 psi and a vertical tensile stress of 300 psi. The shearing stresses are 110 psi and in the directions shown in Fig. 219. Find the maximum resultant stresses and show their planes on free-body diagrams.
Ans. $p = 1{,}510$ psi; $q = 290$ psi; $\tau = 610$ psi.

145-6. In Prob. 145-5, find the stresses on a plane θ making an angle of 30° as shown in Fig. 219,II.

145-7. Find the maximum resultant stresses for an element having a horizontal tensile stress of 860 psi, a vertical compressive stress of 400 psi, and a shearing stress of 160 psi downward on the left side of the element. Sketch the planes of these stresses. *Ans.* $p = 880$ psi.

145-8. An element is subjected to a horizontal tensile stress of 700 psi, a vertical compressive stress of 420 psi, and a shearing stress of 150 psi upward on the left side. Draw Mohr's circle, find the maximum stresses, and locate their planes.

145-9. A 1-in. rod projects upward from a vise, which grips the rod for a length of 1.5 in. The total pressure is 6,000 lb, directed east and west. A pipe wrench is attached to the rod 10 in. above the vise. With the wrench extending east, a horizontal force of 60 lb, directed south, is applied 12 in. from the axis of the rod. Find the resultant shearing and tensile stress at the south surface of the rod at the vise and also at the north surface. Assume that the horizontal compressive stress from the jaws is distributed uniformly over a 1.5-in. length and 1-in. thickness.
Ans. $\tau = 6{,}246$ $p = 7{,}300$ psi, north side
$\tau = 3{,}816$ $q = 8{,}871$ psi, south side

146. Theories of Elastic Failure. In the preceding articles, maximum stresses have been calculated for various types of loading. It is a question, sometimes, what stress determines yield point or incipient failure. It was shown in Chap. 1 that stress in one direction causes deformation in the opposite sense in all directions in any plane perpendicular to the direction of the applied force. It has long been a question whether elastic failure is caused by a stress which has reached a

value greater than the material can withstand or whether the material has been subjected to a critical strain. Several theories[1] have been advanced to explain the cause of elastic failure. A few of these theories follow.

1. *The maximum-stress theory*, called also *Rankine's theory*, states that elastic failure, in any state of complex stresses, will begin when the maximum normal stress (in any direction) equals the elastic limit (stress) which would be obtained in a simple tension (or compression) test. In Fig. 220, if S_t is greater than S_c and the material has an elastic limit in tension equal to that in compression, the maximum normal stress theory says that the material fails elastically in tension when S_t reaches the tensile elastic limit S_e, no matter whether S_c is tension or compression and no matter how large S_c may be, provided, of course, that it is smaller than S_t. If the material is also subjected to shear, as in Fig. 219,I, then elastic failure begins when p in Eq. (145.2) equals S_e.

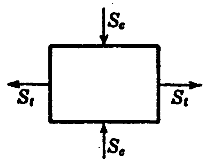


FIG. 220. Tension and compression of right angles.

Cast iron is relatively weak in tension. Figure 87 shows the failure of a cast-iron bar which was tested in torsion. The fracture is at right angles to the line of the maximum resultant tensile stress which was caused by horizontal and vertical shear. When cast-iron bars are subjected to combined bending and twisting, the fracture still takes place at right angles to the maximum resultant tensile stress.

Figure 221 shows a reinforced-concrete beam which was loaded at the third points. Failure began with a diagonal fracture at right angles to the maximum resultant tensile stress. Between the loads, the resultant tensile stress was horizontal, since the shear was practically zero, and the initial cracks were vertical.

While these illustrations show that failure takes place by maximum stress, they prove nothing in regard to the effect of stresses at right angles to the maximum.

2. *The maximum-strain theory*, sometimes called *St. Venant's*[2] *theory*, assumes that a solid reaches its elastic limit when the unit deformation reaches a given limit and that there is an ultimate unit deformation which cannot be exceeded without rupture, no matter in what way the stresses are applied which cause the deformation. In other words,

[1] For an excellent summary of limiting states of stress and theories of strength, see A. Nadai, "Plasticity," McGraw-Hill Book Company, Inc., Chaps. 11–13, pp. 53–74, 1931.

[2] Mariotte seems to have suggested this idea earlier.

when the principal strain, ϵ, reaches the unit deformation obtained at the elastic limit in a tensile test, inelastic action in the material is imminent.

Suppose a block (Fig. 220) is subjected to a direct tensile stress of s_t and to a compressive stress at right angles thereto of s_c, and suppose the material reaches its elastic limit in tension when the unit elongation is 0.001. According to the maximum-strain theory, if the unit elongation which is due to tension is 0.0008 and there is an additional unit elongation of 0.0002 in the same direction which is due to transverse compression, the combined unit elongation of 0.001 brings the material to the elastic limit.

FIG. 221. Failure of a reinforced-concrete beam.

The tensile strength of some materials is much smaller than the compressive strength. If the ratio of the tensile strength to the compressive strength is less than Poisson's ratio for the material, a compressive load should cause failure by transverse tension. This is what seems to happen with porcelain and concrete. A porcelain rod, 1 inch in diameter and 16 inches long, supported a compressive load of 20,000 pounds per square inch and failed by splitting lengthwise. When porcelain is tested in tension, the heads of the specimen must be much larger than the minimum section, or the specimen will fail at the grips. Figure 222 shows the form of a series of bars of rectangular section. The pressure was transmitted to the heads from the grips through leather or lead sheets. Instead of failing at the minimum section, the bars failed along a curved surface AB at one of the heads. According to the maximum-strain theory, the unit deformation across this curved

surface which was caused by tension and the unit deformation which was caused by compression were together greater than the unit deformation in the smaller sections which was caused by tension alone.

3. *The maximum-shear theory*, frequently called the *Guest*[1] *theory* or the *Guest-Hancock*[2] *theory*, states that a material reaches the end of its elastic state when the maximum shearing stress reaches a critical value as determined in a simple tension test. In an ordinary tension test the elastic failure is one of shear occurring on planes running 45° with the axis of the single load and having a magnitude of $\frac{1}{2}S_e$. In Fig. 220 with S_t and S_c acting on the element there is a large resultant shearing stress at 45°. Under certain stress conditions, it is also evident that, when the unit shearing stress reaches the elastic limit in shear, there will be large linear deformations which will appear as the elastic limit in tension or compression. The point upon which there is *not agreement* is whether a solid ever reaches its elastic limit in tension or compression before reaching the elastic limit in shear, and whether failure is always by shear.

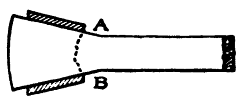


FIG. 222.

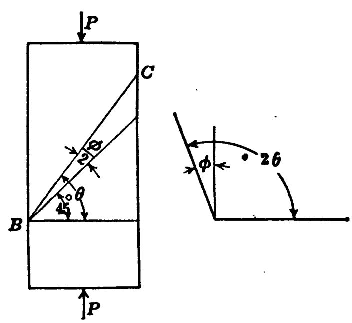


FIG. 223. Shear failure caused by compression.

The tests made by Guest[3] and Hancock were upon ductile materials, and neither of them claimed that the maximum-shear theory applies to brittle solids. In the case of ductile materials, it had been noted that failure does not occur at 45° in a simple tension or compression test. Coulomb[4] suggested that the shearing force on the 45° plane

[1] GUEST, J. J., On the Strength of Ductile Materials under Combined Stress, *Phil. Mag.*, July, 1900, pp. 69–132.

[2] HANCOCK, E. L., The Effect of Combined Stress on the Elastic Properties of Steel, *Proc. ASTM*, 1905, pp. 179–186; 1906, pp. 295–307.

[3] Guest also tested thin hollow cylinders in torsion in an apparent effort to tie the theory to the torsional yield point, but he concluded that "the specific shearing stress at the yield point is better determined by taking one-half of the tensional yield-point stress, than from the results of torsional experiments in which the sharpness of the yield point is masked. . . . " *Ibid.*, p. 132.

[4] Charles Augustin Coulmb (1736–1806), a physicist, was an early experimenter on shear and originated the notion of the application of maxima and minima to a problem in strength of materials.

was assisted by frictional force equal to the normal reaction times a coefficient of internal friction. His theory does check some tests of ductile materials. In Fig. 223, the component of the applied force parallel to BC is $P \sin \theta$ and the component perpendicular to BC is $P \cos \theta$. If A is the area of the normal cross section, the area of the plane through BC normal to the plane of the paper is $A \sec \theta$. If s_s is the unit shearing strength, the shearing resistance is $s_s A \sec \theta$.

When a resolution is taken parallel to BC of Fig. 223, the equation is

$$P \sin \theta = P\mu \cos \theta + s_s A \sec \theta \tag{146.1}$$

in which μ is the coefficient of friction.

$$\frac{P}{s_s A} = \frac{\sec \theta}{\sin \theta - \mu \cos \theta} \tag{146.2}$$

$$\frac{s_s A}{P} = \sin \theta \cos \theta - \mu \cos^2 \theta \tag{146.3}$$

$$\frac{2s_s A}{P} = \sin 2\theta - \mu(1 + \cos 2\theta) \tag{146.4}$$

The load P is a minimum when the second member of Eq. (146.4) is a maximum. When this is differentiated with respect to θ and equated to zero, the result is

$$\cos 2\theta + \mu \sin 2\theta = 0 \tag{146.5}$$

$$\cot 2\theta = -\mu \qquad 2\theta = 90° + \arctan \mu$$

$$\theta = 45° + \frac{\arctan \mu}{2} \tag{146.6}$$

The angle which has a tangent equal to the coefficient of friction is called the *angle of friction* and is generally designated by ϕ.

$$\theta = 45° + \frac{\phi}{2} \tag{146.7}$$

Failure in compression takes place along a plane which makes an angle of 45° plus one-half the angle of friction with the plane normal to the compressive force. Failure in tension takes place at 45°. Timber failure by compression parallel to the grain takes place by buckling the fibers instead of sliding; Eq. (146.7), therefore, does not apply.

4. *The maximum-strain-energy theory*[1] assumes that failure takes place when the energy per unit volume reaches some definite amount which is the energy per unit volume at which a bar of the material will

[1] Suggested by E. Beltrami and developed later by B. P. Haigh.

fail in tension. The energy of tension or compression is $U = s^2/2E$; the energy of shear is $U_s = s_s^2/2G$. If Poisson's ratio is 0.25, $G = 2E/5$, and the energy of unit volume in shear is $U_s = 5s_s^2/4E$.

$$\text{Total } U = \frac{s_t^2}{2E} + \frac{5s_s^2}{4E} \tag{146.8}$$

for combined tension and shear.

The energy may also be expressed as in Eq. (13.9).

5. *The maximum-distortion-energy theory*, sometimes called the *shear-distortion theory*, was developed independently by H. Hencky and R. von Mises and gives results which agree well with tests on some materials. For biaxial stresses the equation resembles the energy equation (13.9).

$$S_e^2 = S_x^2 - S_xS_y + S_y^2 \tag{146.9}$$

where S_x and S_y are of the same sign.

Example

If 40,000 psi is the tensile elastic limit of a steel, 30,000,000 psi is the modulus of elasticity, and ¼ is Poisson's ratio, what is the maximum shearing stress which may be applied in the same plane with a 30,000-psi tensile stress?

1. Solved by *maximum-stress theory*, using Eq. (142.8),

$$S_e = p = \frac{1}{2} S_x + \sqrt{S_s^2 + \left(\frac{S_x}{2}\right)^2}$$
$$40{,}000 = 15{,}000 + \sqrt{s_s^2 + 15{,}000^2}$$
$$s_s = 20{,}000 \text{ psi.}$$

2. According to the *maximum-strain theory*, the critical load is reached when the unit elongation in any direction is equal to the unit elongation which would be caused by the tensile stress of 40,000 psi. Using principal stresses,

$$p = \tfrac{1}{2}S_x + r$$
$$q = \tfrac{1}{2}S_x - r$$

and substituting in the equation for maximum strain,

$$E\epsilon_n = S_e = p - \mu q$$
$$40{,}000 = (1 - \tfrac{1}{4})(15{,}000) + (1 + \tfrac{1}{4}) \sqrt{(15{,}000)^2 + S_s^2}$$
$$S_s = 17{,}400 \text{ psi}$$

3. Solved by the *maximum-shear theory*.

$$20{,}000 = \sqrt{s_s^2 + 15{,}000^2}$$
$$s_s = 13{,}229 \text{ psi}$$

4. For *maximum-strain energy*, substitute in Eq. (146.8)

$$\frac{5s_s^2}{4} = \frac{40{,}000^2}{2} - \frac{30{,}000^2}{2}$$

$$5s_s^2 = 14 \times 10^8 \qquad s_s^2 = 280 \times 10^6$$

$$s_s = 16{,}730 \text{ psi}$$

5. To solve by the *maximum-shear-distortion theory*, substitute the principal stresses in

$$40{,}000^2 = p^2 - pq + q^2$$

$$40{,}000^2 = S_x^2 + 3S_s^2$$

$$40{,}000^2 = 30{,}000^2 + 3S_s^2$$

$$S_s = 15{,}270 \text{ psi}$$

Problems

146-1. Solve the example above if Poisson's ratio is ⅓.

146-2. A given steel has a tensile elastic limit of 36,000 psi. The modulus of elasticity is 30,000,000 psi and Poisson's ratio is ¼. If the material is in simple shear, find the magnitude of the applied unit shearing stress possible by the five theories.

146-3. If the material of Prob. 146-2 is used in a thin hollow cylinder, find the magnitude of the permissible applied normal stresses if the cylinder fails elastically in accordance with the five theories.

146-4. In Prob. 146-3, let the two normal stresses be equal, one tensile and one compressive, and solve.

146-5. A steel plate is subjected to a tensile stress of 16,000 psi and unknown shearing stresses in the same plane. The yield point of the material is 36,000 psi. If $E = 30{,}000{,}000$ and $G = 12{,}000{,}000$ psi, find the permissible shearing stress by applying a factor of safety of 2 to each failure criterion.
Ans. 6,000 psi; 5,310 psi; 4,120 psi; 12,500 psi; and 11,430 psi.

The mathematics of combined stress theory gives no hint about the behavior of materials or tells which of the theories of elastic failure is even approximately correct. Considerable work has been done by a number of investigators in an effort to determine which theory agrees most nearly with tests. None is completely satisfactory, but it has been customary to rely on maximum stress for brittle materials, and many designers use maximum shear for ductile materials, although maximum strain is often used for guns. For some tests maximum energy is very satisfactory, but the recent shear-distortion theory seems even better.[1]

147. Some Other Factors Affecting Failure. Experience shows that an automobile spring will often fail in service after repeated

[1] For a thorough discussion see Joseph Marin, "Mechanical Properties of Materials and Design," McGraw-Hill Book Company, Inc., New York, 1942. See also C. Richard Soderberg, Working Stresses, *Trans. ASME*, Vol. 55, A PM 55-16, pp. 131–144, 1933.

flexure although the maximum stress developed is less than the proportional limit of the material. Many automotive, airplane, steam turbine parts, etc., give similar trouble. The earliest experiments of this phenomenon were made by Wöhler about 1860 on axles of the German railways. This type of failure was called *fatigue.*

When steel fails under repeated applications of load, the fracture has a crystalline appearance. For this reason it was long thought that repeated stresses cause the formation of crystals in the steel. Microscopic[1] examination shows that all steel is crystalline and that crystals do not form at atmospheric temperatures. The crystalline appearance of the fracture is due to the fact that the fracture has taken place across the crystals of the steel. Since materials are not homogeneous and isotropic, as assumed in deriving stress-strain relations, the properties vary from point to point with minute discontinuities or flaws between the crystals.

Fatigue tests may embrace direct stress, bending, torsion, or any combination of them. The maximum stress which a material will stand in spite of an infinite number of complete reversals of stress is called the *endurance limit.* It is obtained by testing specimens at different stresses and noting the number of cycles of repeated applications of load which cause failure. For instance, steel of 0.49% carbon which was tested by Moore and Jasper[2] had an ultimate tensile strength of 91,500 pounds per square inch and a proportional elastic limit of 44,700 pounds per square inch. When this steel was tested under complete reversal of stress, the *endurance limit* was found to be 33,000 pounds per square inch. If the unit stress was changed from 33,000 pounds per square inch tension to 33,000 pounds per square inch compression, the specimen would stand an indefinite number of repetitions without failure. Two specimens of this material were each subjected to over 100 million repetitions of a stress which varied from 33,100 pounds per square inch tension to 33,100 pounds per square inch compression. Neither of these failed. A third specimen was subjected to 1 billion repetitions of a stress which varied from 33,000 pounds per square inch tension to the same compression without failure. The stress of 33,000 pounds per square inch is, therefore given as the *endurance limit* of this steel for complete reversal of stress. When the stress in this material

[1] For a complete list of papers on the fatigue of metals, see *Univ. Ill. Eng. Expt. Sta. Bull.* 124, pp. 168–178. See also H. F. Moore and J. B. Kommers, "The Fatigue of Metals," McGraw-Hill Book Company, Inc., New York, 1927.

[2] MOORE, H. F., and T. M. JASPER, *Univ. Ill. Eng. Expt. Sta. Bull.* 136, pp. 23 and 31.

varied from 34,000 to −34,000 pounds per square inch, each of the two specimens tested failed at about 4 million repetitions. At 37,000 pounds per square inch, the failure was at 1,225,000 repetitions; and at 50,000 pounds per square inch, the failure was at 42,000 repetitions.

If the stress is not completely reversed, the endurance limit is higher. The smaller the range between the maximum and minimum stress, the greater is the maximum stress which the material will endure for an indefinite time. The carbon steel above mentioned[1] has an endurance limit of 33,000 pounds per square inch for complete reversal of stress. The same material stood an indefinite number of repetitions when the

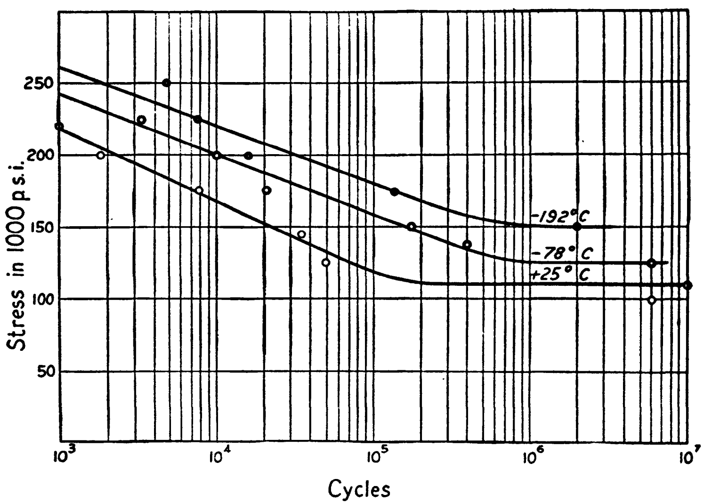


FIG. 224. S-N diagram for stainless steel.

stress ranged from 36,000 pounds tension to 21,600 pounds compression, from 47,000 pounds tension to zero, from 60,000 pounds tension to 12,000 pounds tension, or from 69,000 pounds tension to 34,500 pounds tension. These tests and many others show that the endurance is higher the smaller the range of stress.

Results of tests are plotted on a diagram, called the S-N *diagram.* Data for stainless steel at three temperatures are plotted in Fig. 224. The tests were made on polished specimens 0.300 inch in diameter on a reciprocating-beam machine at 2,000 revolutions per minute with reversal of stress.[2] Temperature has considerable effect on all alloys, the fatigue strength increasing with decreasing temperature. Other

[1] *Univ. Ill. Eng. Expt. Sta. Bull.* 136, p. 77.

[2] Tested by M. G. Fontana, research professor, and J. L. Zambrow, research engineer, at The Ohio State University Engineering Experiment Station.

properties are also affected by changes in temperature, as for instance Charpy impact, Vickers hardness, modulus of elasticity, ultimate strength, yield strength, and ductility. Much information is still needed on low-temperature behavior. Failure of metals at high temperatures is known as *creep*. The question facing the engineer is the amount of plastic deformation which can be tolerated. Often the fatigue strength is more significant than the creep rate.

A number of methods have been proposed for designing members which are subjected to variable loads. This may be done by reducing the allowable unit stress or by adding an increment to the applied load.

Professor John Goodman[1] suggested a rule for this purpose. Goodman's "dynamic" rule is *add to the maximum load the difference between the maximum and minimum loads and treat the sum as a static load.* Under this rule, a load which varied from 12,000 pounds tension to 12,000 pounds compression would be equivalent to

$$12{,}000 + 24{,}000 = 36{,}000 \text{ pounds tension or compression}$$

a load which varied from 12,000 pounds tension to 6,000 pounds compression would be equivalent to

$$12{,}000 + 18{,}000 = 30{,}000 \text{ lb tension}$$

and a load which varied from 12,000 pounds tension to 4,000 pounds tension would be equivalent to 20,000 pounds tension.

Moore and Jasper have shown[2] that Goodman's dynamic rule errs on the side of safety. They suggest the following formula, which is the equation of a straight line,

$$S'_r = S_r\left(\frac{r+3}{2}\right)$$

in which r is the ratio of the minimum stress to the maximum stress, S'_r is the endurance limit for the ratio r, and S_r is the endurance limit for complete reversal. When the stress is reversed, r is negative. For instance, if the stress changes from 8,000 pounds per square inch compression to 20,000 pounds per square inch tension, r is -0.4. To use this formula, of course, the endurance limit for complete reversal must be determined experimentally. Moore and Jasper state further:[3]

[1] GOODMAN, "Mechanics Applied to Engineering," 4th ed. Longmans, Roberts and Green, London, 1904, p. 535.

[2] *Univ. Ill. Eng. Expt. Sta. Bull.* 136, pp. 82–89.

[3] *Ibid.*

This formula can be used only up to the limit at which the maximum unit stress set up reaches the proportional elastic limit of the material, and for most steels this eliminates ratios of minimum stress to maximum stress greater than zero. Beyond the proportional elastic limit the static properties of the steel are the governing factors rather than the fatigue properties.

Other rules estimating the endurance limit for other ranges of stress include Gerber's rule and various straight-line equations. If the ratio of the endurance limit for complete reversal to the elastic limit of the material is called m,

$$S'_r = (1 + m)S_{\text{ave}} + S_r$$

where S_{ave} is the average stress during the cycle.

In Art. 31 the effect of form of the specimen on the unit stress was discussed. When the hole is in the middle of a very wide plate (Fig. 51) the maximum stress is approximately three times the average. If the hole is moved close to one edge the stress concentration factor may be over 4.0. If the plate is narrow so that the diameter of the hole is nearly three-quarters the width of the plate, the stress concentration factor may be 8. Notches cut in the sides of a plate as in Fig. 52,II may run the stress up to twenty times the average.

Stress concentrations on fatigue specimens will likewise reduce the endurance limits. As an example, a rough finish instead of a well-polished one will cut the endurance limit 20 per cent. The effect of corrosion on fatigue may be serious. Merely running water on the specimen during the tests will reduce the fatigue strength 20 to 40 per cent.

148. Miscellaneous Problems

148-1. A 6- by 8-in. beam of dense southern yellow pine is 5 ft long between supports and carries 6,000 lb 2 ft from the left support. Find the maximum unit compressive stress. Find the unit compressive stress and the unit shearing stress 0.8 in. from the top at a section 20 in. from the left support. Find the maximum unit compressive stress at this point.

148-2. What is the average vertical shearing stress in any section to the left of the load for the beam of Prob. 148-1? What is the shearing stress at the neutral surface? *Ans.* 75 psi; 112.5 psi.

148-3. A hollow shaft of 4 in. inside diameter and 6 in. outside diameter is subjected to a torque of 2,000 ft-lb and a bending moment of 1,500 ft-lb. Find the equivalent maximum torque and moment and find the maximum unit shearing and tensile stress.

148-4. A 3-in. standard pipe is subjected to a bending moment of 13,792 in.-lb and a torque of 20,688 in.-lb. Find the maximum tensile and shearing stresses. *Ans.* p = 11,201 psi; τ = 7,205 psi.

CHAPTER 16

SPECIAL BEAMS

149. Beams of Constant Strength. In a beam of "constant strength" the unit stress in the outer fibers is the same at all sections. Since $S = Mc/I = M/Z$, the stress is constant when the section modulus varies as the bending moment. In a cantilever with a load on the end, for instance, the moment is directly proportional to the distance from the end. If the depth is constant and the width increases uniformly from the free end to the fixed end, the section modulus varies directly as the moment, and the unit stress in the outer fibers is constant. If it were not necessary to make some al owance for shear and compression at the free end, this beam would be only one-half as heavy as a uniform beam of equal strength. Even with the additional material to meet the requirements of shear and compression, a great saving in weight is secured by the use of "beams of constant strength."

150. Cantilever with Load on the Free End. With the origin of coordinates at the free end of the cantilever, the moment at a distance x from the end is Px. (It is not necessary to consider the sign of the moment, since the unit stress depends upon the magnitude only.) If S is the allowable bending stress and Z is the section modulus,

$$Px = SZ$$

Since the section modulus for a rectangular section is $bd^2/6$,

$$Px = \frac{Sbd^2}{6}$$

Problems

150-1. A cantilever of rectangular section, with a load P on the free end, has a constant depth of 5 in. If the allowable stress in the outer fibers at any section is 1,200 psi, what is the equation for the breadth in terms of the load on the end and the distance x from the end? *Ans.* $b = Px/5{,}000$.

150-2. A cantilever of rectangular section, with the load on the free end, has a constant breadth b. What is the expression for the depth at any section at a distance x from the load? *Ans.* $d^2 = 6Px/Sb$.

150-3. A rectangular cantilever of constant strength, 5 ft long, with a load of 600 lb on the free end, has a constant breadth of 4 in. Derive an expres-

sion for the depth at any section for an allowable unit stress of 1,000 psi and calculate the depth for intervals of 10 in. *Ans.* $d^2 = 0.9x$.

x	10	20	30	40	50	60
d^2	9	18	27	36	45	54
d, in.	3	4.24	5.20	6.00	6.71	7.35

150-4. A cantilever of constant strength and square section, 5 ft long, carries a load of 720 lb on the free end. Derive an expression for the depth of any section for a unit stress of 1,200 psi. Arrange the work as in Prob. 150-3 and calculate the depth at 12-in. intervals. *Ans.* $d^3 = 3.6x$.

150-5. A cantilever of constant strength, loaded at the free end, has all sections similar rectangles. The breadth is 4 in. and the depth is 8 in. at 80 in. from the load. Derive an expression for the depth in terms of the distance from the load and calculate the breadth at each 10-in. interval.
Ans. $d^3 = 6.4x$; $b = 3.42$ in. at 50 in. from load.

150-6. A cantilever of constant strength and circular section carries a load of P lb on the free end. Derive the expression for the diameter at any section in terms of the allowable stress and the distance from the load.
Ans. $d^3 = 32Px/\pi S$.

150-7. A cantilever of constant strength and circular section throughout is 4½ ft long and carries 3,142 lb on the free end. If the allowable bending stress is 1,000 psi, find the size at 9-in. intervals.

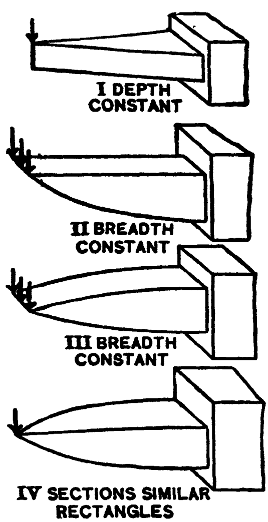


FIG. 225. Cantilevers of constant strength.

151. Shearing and Bearing Stresses at the End. In Fig. 225, the load P is represented at the extreme ends of the beams. Allowance must be made at the ends for the bearing and shearing stresses. For instance, in Prob. 150-3, suppose the allowable unit shearing stress to be 150 pounds per square inch. The *average* unit shearing stress in a rectangular section will be 100 pounds per square inch and the minimum area of cross section will be 6 square inches. The depth at the end should not be less than 1.5 inches.

Suppose also that the allowable bearing stress is 300 pounds per square inch, and that the *center* of the load must be 5 feet from the wall; the bearing area must be at least 2 square inches. If the load extends the entire width of the beam, the bearing area must be 4 by ½ inch. The actual beam must extend at least ¼ inch beyond the center of the load. Figure 226 shows the details for these conditions.

The dotted lines are the limits for the beam figured for bending only. The solid lines show the *minimum* dimensions figured for all stresses. The actual beam should be somewhat larger at the end than shown, as a great increase in safety can be secured here with practically no increase in cost and weight. Artistic appearance and convenience of

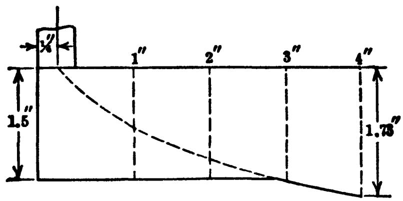


FIG. 226. Beam of constant strength.

construction may cause further modifications *outside* the minimum dimensions.

Problems

151-1. Design a cantilever of constant strength and constant depth of 4 in. to carry a load of 600 lb 4 ft from the fixed end. The allowable bending stress is 1,200 psi; the allowable bearing stress is 240 psi; and the allowable shearing stress parallel to the grain is 180 psi.
Ans. Maximum width, 9 in.; minimum width, 1.25 in.; minimum bearing surface extends 1 in. on each side of the line of application of the load.

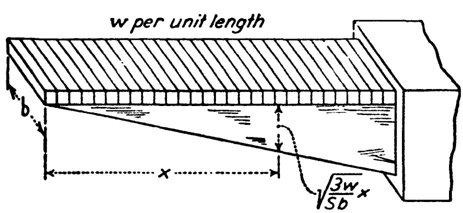


FIG. 227. Cantilever of constant breadth.

151-2. A cantilever of constant strength and constant breadth of 4 in. is designed to carry 900 lb 60 in. from the fixed end. The allowable bending stress is 1,000 psi, the allowable bearing stress is 225 psi and the allowable horizontal shearing stress is 150 psi. Design the beam.
Ans. Maximum depth, 9 in.; minimum depth, 2.25 in.; beam extends 60.5 in. from the fixed end.

152. Cantilever with Uniformly Distributed Load. The only difference between a cantilever with uniformly distributed loading and a cantilever with a concentrated load on the end is in the expression for the external moment.

Problems

152-1. A cantilever of constant strength, which carries a uniformly distributed load of w per unit length, has a constant breadth b. Derive the expression for the depth at any distance x from the free end.

Ans. $d^2 = 3wx^2/Sb$.

152-2. A cantilever of constant strength and constant breadth of 4 in. carries a uniformly distributed load of 240 lb per ft. Find the depth at 4 ft from the free end if the allowable stress is 960 psi. *Ans.* 6 in.

152-3. A uniformly loaded cantilever of constant strength has a constant depth d. If all sections are rectangles, find the expression for the breadth (Fig. 228). *Ans.* $b = 3wx^2/Sd^2$.

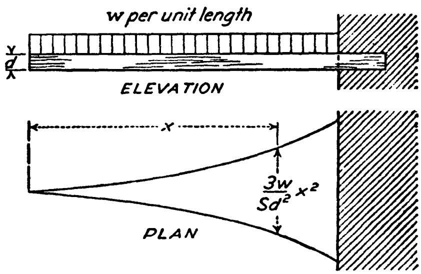


FIG. 228. Cantilever of constant depth.

152-4. A cantilever of constant strength, uniformly loaded, is 5 ft long and 4 in. deep. The maximum breadth is 8 in. Find the breadth at each foot interval. If the allowable bending stress is 1,080 psi what is the total load?

Ans. $b = \dfrac{3w}{Sd^2}x^2 = Kx^2 \qquad 8 = 3{,}600K \qquad K = \dfrac{1}{450}$

x, ft............	1	2	3	4	5
b, in............	0.32	1.28	..	..	8

$W = 768$ lb; $w = 12.8$ lb per in.

152-5. If the allowable horizontal shearing stress of the beam of Prob. 152-4 is 120 psi, for what distance from the free end must the beam be designed for shear, and what is the form of this portion?

Ans. Total vertical shear per inch of width $= 80 \times 4$

$$320b = \frac{320x^2}{450} \qquad \frac{320x^2}{450} = 12.8x \qquad x = 18 \text{ in.}$$

The first 18 in. is a triangle which is 0.48 in. wide at 1 ft from the end.

NOTE: No provision is made for bearing in this design.

152-6. A cantilever of constant strength and constant depth of 4 in. is 6 ft long. It carries 120 lb per ft and 300 lb 1 ft from the free end. If the bending stress is 1,200 psi, find the breadth at each foot interval.

Ans.

x, ft.	1	2	3	4	5	6
M, in-lb	720	6,480				
Z, in^3	0.6	5.4				
b, in	0.225	....	4.275	..	10.125	

152-7. The beam of Prob. 152-6 is 6 in. wide throughout. Find the depth at each foot interval.
Ans.

x, ft.	1	2	3	4	5	6
d, in.	0.775	2.324	..	4.313	..	6.050

152-8. Derive an expression for the depth of a cantilever of square section to carry a uniformly distributed load. *Ans.* $d^3 = 3wx^2/S$.

152-9. A uniformly loaded beam of constant strength and square section carries a load of 9 lb per in. The allowable stress is 1,000 psi. Find the dimensions at 64 in. from the free end. *Ans.* $d = 4.8$ in.

152-10. A cantilever which carries 9 lb per in. has all sections similar rectangles for which the depth is twice the breadth. A load of 180 lb is placed 10 in. from the free end. Find the dimensions for 10-in. intervals up to 5 ft if $S = 1{,}000$ psi.
Ans.

x, in.	10	20	30	40	50	60
d, in.	1.754	3.509	..	..	6.050	

152-11. How does the volume of the beam of Prob. 152-1 compare with that of a uniform beam of equal strength? *Ans.* One-half as great.

152-12. How does the volume of the beam of Prob. 152-3 compare with that of a uniform beam of equal strength if no correction is made for shear? *Ans.* One-third as great.

152-13. How does the volume of the beam of Prob. 152-5 compare with that of a uniform beam of equal strength? *Ans.* 33.78 per cent.

153. Beam of Constant Strength, Simply Supported. When a beam is supported at the ends and carries a single load at the middle, the problem is exactly the same as that of a cantilever of one-half the length which is pushed up by the end reaction. When the load is not at the middle, the portion from the load to each end is equivalent to a cantilever. Ample allowance must be made for shear and bending at the supports.

A uniformly loaded beam which is simply supported is equivalent to cantilevers which are pushed up by the end reactions and bent down by the distributed loads. Since the moment equation is composed of two terms, the calculations are not so simple as for a cantilever.

Problems

153-1. A cast-steel beam is made for a span of 8 ft to carry a distributed load of 2,400 lb per ft with a maximum unit stress of 12,000 psi. Find the section modulus at each 12 in.
Ans.

x, ft.	1	2	3	4
Z, in.^{3}	8.4	14.4	18	19.2

153-2. A timber beam of constant strength and constant breadth of 8 in. is used for a span of 10 ft to carry a load of 240 lb per ft with a maximum stress of 1,200 psi. Find the depth at each foot.
Ans.

x, ft.	1	2	3	4	5
d^2, in.^{2}	8.1	14.4	18.9	21.6	22.5
d, in.	2.848	3.795	4.347	4.648	4.743

153-3. Derive the expression for the depth of a uniformly loaded, simply supported beam of constant breadth. *Ans.* $d = \sqrt{3w(lx - x^2)/Sb}$.

Example

A 20-in. 95-lb I beam is used for a span of 30 ft to carry 180,000 lb uniformly distributed. The beam is reinforced by 12- by 1-in. plates welded to the top and bottom flanges. If the maximum bending stress is 15,000 psi, find the minimum length of each pair of plates.

For the first pair of plates,

$$90{,}000x - 250x^2 = 15{,}000 \times 160$$
$$x^2 - 360x + 9{,}600 = 0$$
$$x = 29 \text{ in.}$$

Each of the first pair of plates is $360 - 58 = 302$ in. in length.

The additional moment of inertia is $\dfrac{12(22^3 - 20^3)}{12}$.

$$I = 2{,}648 + 1{,}600 = 4{,}248 \text{ in.}^4$$
$$Z = \frac{4{,}248}{11} = 386.2 \text{ in.}^3$$
$$x^2 - 360x + 23{,}172 = 0 \qquad x = 84 \text{ in.}$$

Each of the second pair of plates is 16 ft long. Will another pair of plates be required?

Shapes are not rolled as beams of constant strength, but combinations of shapes and plates, as in the example, are riveted or welded together in such a way that the section modulus varies approximately as the bending moment. In machinery and vehicles, where weight is important, beams of constant strength are much used. In the frames

of stationary machines, these are frequently made of cast iron. In other places, cast steel or forged steel is employed. Cast-steel members, of approximately constant strength, are used in the construction of railway cars and trucks, and steel forgings in automobiles.

A tree is a vertical beam of constant strength. A bamboo rod or a wheat straw is a hollow beam of constant strength, which has a large section modulus relative to its weight.

Problems

153-4. An 8-in. 23-lb standard I beam is used to support 8,000 lb per ft on a span of 8 ft. The beam is reinforced with 4- by 1-in. plates welded to the top and bottom flanges. If the maximum allowable bending stress is 16,000 psi, find the length of plates required. How many sets of plates are required?

153-5. A 10-in. 25.4-lb standard I beam is reinforced with 5- by 1-in. plates on the top and bottom flanges were needed. The beam is supported at the ends on a 12-ft span and carries 7,000 lb per ft uniform load. Find the length of each set of plates required, if the maximum bending stress is 18,000 psi.

153-6. A 7-in. 20-lb standard I beam is supported at the ends on a 12-ft span. It is reinforced by flange plates over a part of its length, each plate being 4- by 1 in. If the uniform load is 4,000 lb per ft and the allowable bending stress is 18,000 psi, find the length of each plate.

154. Deflection of Beam of Constant Strength. Since the moment of inertia is not constant, the fundamental equation for successive integration is written

$$E\frac{d^2y}{dx^2} = \frac{M}{I}$$

and the equation for area moments is written

$$Ey = \int \frac{M}{I} x\,dx$$

The moment of inertia is expressed in terms of x before integration.

In the following derivations, the maximum moment of inertia is written I_m. Constant depth and depth at the maximum section are expressed by D; and constant breadth and breadth at the maximum section are expressed by B.

When the depth is constant, the moment of inertia and the breadth vary directly as the section modulus, which, in turn, varies directly as the moment. For all beams of constant strength and constant depth, the calculations are the same as for a beam of constant moment, *provided the breadth of every surface parallel to the neutral surface varies directly as the moment.*

155. Cantilever of Constant Depth. For a cantilever of constant strength and constant depth, loaded at the free end,

$$I = \frac{I_m x}{l} \qquad M = -Px \qquad \frac{M}{I} = -\frac{Pl}{I_m}$$

$$Ey = -\frac{Pl}{I_m}\int_0^l x\,dx = -\frac{Pl^3}{2I_m}$$

The deflection at the end is $1\frac{1}{2}$ times as great as the deflection of a beam of constant section for which $I = I_m$. To find the deflection at a distance a from the end by area moments,

$$Ey = -\frac{Pl}{I_m}\int_a^l (x-a)\,dx = -\frac{Pl}{2I_m}\Big[(x-a)^2\Big]^l = -\frac{Pl}{2I_m}(l-a)^2$$

Example

A cantilever of constant depth d, loaded at the free end, has a constant section of width Ba/l for a distance a from the load. The remainder of the beam is designed for constant strength. Find the expression for the deflection at the free end by graphic integration of area moments.

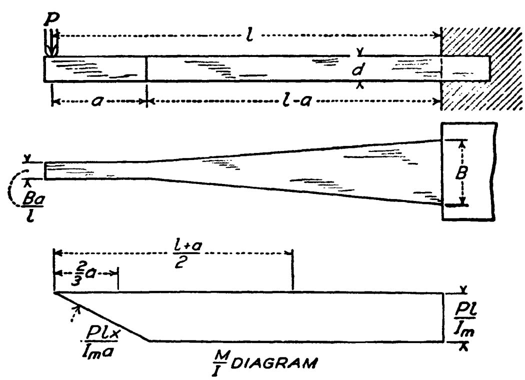


FIG. 229. Cantilever of constant depth with provision for shear.

The M/I diagram is shown in Fig. 229. The diagram for the uniform portion of length a is a triangle. The remainder of the diagram, of length $l - a$, is a rectangle. The deflection at the end is given by

$$Ey = -\frac{Pl}{I_m}\times\frac{a}{2}\times\frac{2a}{3} - \frac{Pl}{I_m}(l-a)\frac{l+a}{2} = -\frac{Pl}{I_m}\left(\frac{l^2}{2} - \frac{a^2}{6}\right)$$

Problems

155-1. A cantilever 6 ft long has triangular sections which are 6 in. deep throughout. The first 18 in. has a constant width of 2 in. at the top. The remainder is designed as a beam of constant strength for a load 6 ft from the fixed end. The maximum width is 8 in. When the load is 200 lb,

find the maximum stress. Find the deflection at the free end if the modulus of elasticity is 1,000,000 psi

Ans. $S = 1{,}200$ psi; $y_{max} = 0.7614$ in.

155-2. If the beam of Prob. 155-1 has elliptical sections with all vertical axes 6 in. and the maximum horizontal axis 8 in., with horizontal axes of 2 in. for the first 18 in., while the remainder is designed for constant strength, find the maximum stress, the deflection at the end, and the deflection 2 ft from the free end. *Ans.* $y_{max} = 0.4309$ in.; $S = 509.3$ psi.

155-3. The beam of Prob. 155-2 has a circular section at 54 in. from the free end. Calculate the stress at this section.

For a uniformly loaded cantilever of constant depth,

$$I = \frac{I_m x^2}{l^2} \qquad M = -\frac{wx^2}{2} \qquad \frac{M}{I} = -\frac{wl^2}{2I_m}$$

$$Ey_{max} = \frac{wl^2}{2I_m}\int_0^l x\,dx = -\frac{wl^4}{4I_m}$$

which is twice the maximum deflection of a cantilever of uniform section, similarly loaded.

To find the deflection at any distance a from the free end,

$$Ey = -\frac{wl^2}{2I_m}\int_a^l (x - a)\,dx = -\frac{wl^2(l-a)^2}{4I_m}$$

Problems

155-4. Find the maximum deflection and the deflection 18 in. from the free end for the beam of Prob. 152-4 if $E = 1{,}200{,}000$ psi.

Ans. 0.81 in.; 0.3969 in.

155-5. Find the deflection at the free end of the beam of Prob. 152-5 if the modulus of elasticity is 1,200,000 psi.

For the triangle 18 in. long,

$$I = \frac{0.64x}{3} \qquad \frac{M}{I} = -30x$$

$$Ey_{max} = -30\int_0^{18} x^2\,dx - 540\int_{18}^{60} x\,dx = -58{,}320 - 884{,}520 = -942{,}840$$

$$y_{max} = -0.7857 \text{ in.}$$

155-6. Solve Prob. 155-4 by means of the moment of the area of the M/I rectangle.

155-7. Solve Prob. 155-5 by means of the area of the M/I rectangle for 18 to 60 in. and the triangle from 0 to 18 in.

156. Simply Supported Beam of Constant Depth. *For a simply supported beam of constant strength and constant depth, loaded at the middle,* the maximum moment at the middle is $Pl/4$, and $M/I = Pl/4I_m$.

The deflection of the end upward from the tangent at the middle by area moments from the M/I rectangle is

$$\frac{Pl}{4I_m} \times \frac{l}{2} \times \frac{l}{4} = \frac{Pl^3}{32I_m} = Ey_{max}$$

The slope at the left support is given by

$$0 = E\theta_1 + \frac{Pl^2}{8I_m} \qquad E\theta_1 = -\frac{Pl^2}{8I_m}$$

$$Ey = -\frac{Pl^2x}{8I_m} + \frac{Plx}{4I_m} \times \frac{x}{2} = -\frac{Plx}{8I_m}(l - x)$$

For a simply supported beam of constant strength and constant depth, uniformly loaded, $M/I = wl^2/8I_m$.

At the middle,

$$0 = E\theta_1 + \frac{wl^3}{16I_m} \qquad E\theta_1 = -\frac{wl^3}{16I_m}$$

$$Ey = -\frac{wl^3x}{16I_m} + \frac{wl^2x}{8I_m} \times \frac{x}{2} = -\frac{wl^2x}{16I_m}(l - x)$$

At the middle, where $x = l/2$,

$$Ey = -\frac{wl^4}{64I_m}$$

which is six-fifths as great as the maximum deflection of a beam of constant section.[1]

157. Beams of Two or More Materials. Beams are frequently made of two or more materials which have different moduli of elasticity.

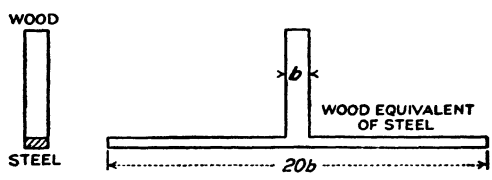


FIG. 230. Beam of two materials.

The most common types are combinations of timber and steel or of concrete and steel.

Figure 230 shows a steel plate which is bolted to the bottom of a wooden beam. The position of the neutral surface depends upon the ratio of the moduli of elasticity of the two materials. If the modulus of the steel is 30,000,000 pounds per square inch and that of the wood is 1,500,000 pounds per square inch, the unit stress in the steel at a given unit deformation is twenty times as great as the unit stress in the wood.

[1] Some other cases were discussed in the 4th edition.

The location of the neutral axis for Fig. 230 may be computed on the assumption that the density of the steel is twenty times as great as the density of the wood; or, for purpose of calculation, the steel may be replaced by a wooden strip which is twenty times as wide and has the same thickness. Figure 230,II, illustrates this substitution.

Example

A 4- by 6-in. wooden beam has a steel plate 1 in. wide and ½ in. thick fastened to the lower surface. Find the neutral axis and the maximum fiber stress in each material if the modulus of elasticity of the steel is twenty times as great as that of the wood, and the bending moment is 30,000 in.-lb.

The steel may be replaced by a wooden strip 20 in. wide and ½ in. thick. To get the distance of the center of gravity from the bottom of the wood,

$$\bar{y} = \frac{24 \times 3 - 10 \times \frac{1}{4}}{34} = 2.04 \text{ in.}$$

To get the moment of inertia of the equivalent wooden section about the common surface,

$$\frac{4 \times 6^3}{3} = 288$$

$$\frac{20 \times (\frac{1}{2})^3}{3} = \quad 0.83$$

$$I = 288.83$$

$$I_c = 288.83 - 34 \times 2.04^2 = 147.34 \text{ in.}^4$$

To get the unit stress in the top fibers of the wood,

$$S = \frac{30{,}000 \times 3.96}{147.34} = 806 \text{ psi}$$

In the bottom steel fibers,

$$S = \frac{30{,}000 \times 2.54 \times 20}{147.34} = 10{,}344 \text{ psi}$$

The result for steel is multiplied by 20 because the moment of inertia used was calculated on the assumption that the steel was replaced by wood.

Problems

(*Use E for steel, twenty times E for timber, in these problems.*)

157-1. A 4- by 4-in. timber beam has a 4- by ½-in. steel plate on the lower surface and a 2- by 1-in. plate on the upper surface. Find the neutral axis of the combination. What is the maximum fiber stress in the steel when that in the wood is 600 psi.?
Ans. Neutral axis, 2.10 in. above bottom of timber; fiber stress in steel, 16,571 psi.

157-2. A 6- by 6-in. timber beam, 10 ft long, has a 6- by ½-in. steel plate on the top and bottom surfaces. Find the unit stress in the steel when a load of 9,000 lb is put on the middle. *Ans.* 13,720 psi.

157-3. A 4- by 6-in. timber beam has a 4- by ½-in. steel plate on the top surface and a 4- by 1-in. plate on the bottom. The beam is supported at the ends on a 10-ft span and carries 40,000 lb at 3 ft from the left end, and 60,000 lb at 4 ft from the right end. Find the maximum bending stress in each material.

158. Reinforced-concrete Beams. Reinforced concrete represents another form of combination beam. A reinforced-concrete beam has steel rods embedded in the concrete near the surface in the tension side. Sometimes both tension and compression sides are reinforced. These rods may be ordinary round or square steel bars. Usually they are corrugated or otherwise deformed or made of cable or twisted square bar. Such deformed or twisted bars are better fitted to transmit the stress from the concrete, since they do not slip when the grip of the concrete is weakened.

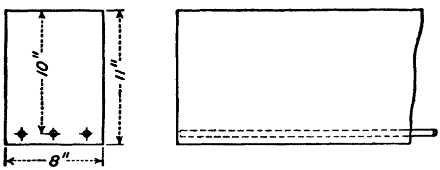


FIG. 231. Reinforced-concrete beam.

Figure 231 represents a section and a portion of the length of a reinforced-concrete beam, which is 8 inches wide and 11 inches deep. Figure 220 is a photograph of a beam of this size after failure. The reinforcement consists of three rods with centers 1 inch from the bottom of the concrete.

In the development of the theory of concrete beams, it is customary to consider that the steel takes all the tension. This assumption is correct for loads commonly used in practice. For small loads, giving tensile stresses of less than 300 pounds per square inch in the concrete (depending upon the quality), some of the tension is carried by the concrete. For larger loads, fine cracks form in the tension side, and experiments show that the steel takes practically all tension.

The *percentage of reinforcement* in a beam is calculated by dividing the area of the steel by the area of the beam section above the center of the steel. In Fig. 231 the beam is regarded as an 8- by 10-inch section; the inch of concrete below the center of the rods is considered as simply protecting the steel. With three ⅝-inch rods, each of which has a cross section of 0.307 square inch, the reinforcement in the beam of Fig. 231 is $0.921/80 = 0.0115 = 1.15$ per cent. While it is customary to speak of the *percentage of reinforcement,* when used in formulas it is expressed as a ratio.

A Joint Committee from the American Society of Civil Engineers, the American Society for Testing Materials, the American Railway Engineering Association, and the Association of American Portland Cement Manufacturers has prepared a report on Concrete and Reinforced Concrete and has recommended certain formulas and constants. In the articles that follow, the important formulas are given in the form recommended by the Joint Committee, and with the same symbols, including the use of f for unit stress instead of the usual S. Subscripts denote the material.

The line OF of Fig. 232 represents the compressive stress. The depth from the extreme compressive fibers to the center of the rein-

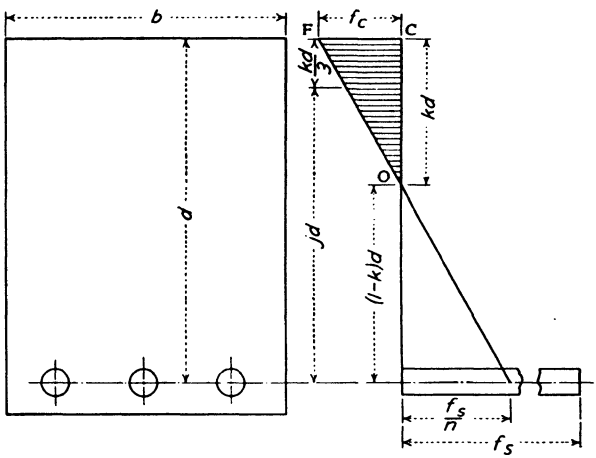


FIG. 232. Stress in rectangular reinforced-concrete beam.

forcement is d and the distance from the neutral surface to the extreme fibers is kd, in which k is a fraction less than unity. The distance from the neutral surface to the center of the reinforcement is $(1 - k)d$. The ratio of the modulus of elasticity of the steel to that of the concrete is represented by n.

$$n = \frac{E_s}{E_c} \tag{158.1}$$

If the modulus of elasticity of the steel is 30,000,000 and that of the concrete in compression is 2,000,000 pounds per square inch, the value of n is 15. This is the value which was formerly in general use. However, when tests show the ultimate strength of the concrete to be between 2,500 and 2,900 pounds per square inch, the Joint Committee recommends that n should be 12; and for concrete 3,000 pounds per

square inch or stronger, the committee recommends that n should be 10.

The unit deformation at the center of the reinforcement is to the unit deformation in the outer fibers of the concrete as $1 - k$ is to k. Since the modulus of the steel is n times the modulus of the concrete, the unit stress at the center of the reinforcement is

$$f_s = \frac{n(1 - k)}{k} f_c \tag{158.2}$$

in which f_s is the unit tensile stress at the center of the steel reinforcement, and f_c is the maximum unit compressive stress in the concrete. The tensile stress at the center of the reinforcement is the average tensile stress, and it is assumed that the resultant tensile stress is at the center of the section. (The error is negligible.) The area of the concrete in compression in a rectangular section is bkd, and the average unit stress over this area is $f_c/2$.

$$\text{Total compressive force} = \frac{f_c kbd}{2} \tag{158.3}$$

$$\text{Total tensile force in steel} = \frac{Anf_c(1 - k)}{k} \tag{158.4}$$

in which A is the area of the steel. The ratio of the area of the steel to the area of the concrete is represented by p.

$$p = \frac{A}{bd} \qquad A = pbd \tag{158.5}$$

As the concrete below the neutral surface is not regarded as taking any of the tensile stress, the total tension in the steel equals the total compression in the concrete. Equating (158.3) and (158.4) and substituting for A,

$$\frac{f_c kbd}{2} = \frac{f_c pbdn(1 - k)}{k} \tag{158.6}$$

$$k^2 = 2pn(1 - k) \tag{158.7}$$

$$k^2 + 2pnk - 2pn = 0 \tag{158.8}$$

$$k = \sqrt{2pn + (pn)^2} - pn \tag{158.9}$$

Problems

158-1. If the modulus of the steel is taken as fifteen times that of the concrete and the area of the steel is 1 per cent of the total area bd, find the distance of the neutral axis from the extreme compression fibers. *Ans.* $k = 0.418$.

158-2. Solve Prob. 158-1 for a reinforcement of 1.2 per cent and for 1.6 per cent. *Ans.* 0.446, 0.493.

158-3. If $n = 12$, find k for 1 per cent reinforcement.

158-4. If $n = 10$, find k for 1 per cent reinforcement.
158-5. If $n = 10$, find k for 1.2 per cent reinforcement.

159. Stresses in Concrete and Steel. When the location of the neutral axis has been determined, the relative values of the average unit compressive stress in the concrete and the average unit tensile stress in the steel may be computed from the relation that the total tension in the steel is equal to the total compression in the concrete.

The Joint Committee on Concrete recommends as the maximum allowable working stress in the extreme fibers in bending a value of 0.45 of the ultimate compressive strength of 28-day concrete test cylinders. Hence, for what is called 2,000-pound concrete, the allowable $f_c = 900$ pounds per square inch; for 3,000-pound concrete, $f_c = 1{,}350$ pounds per square inch. The allowable stress in the steel is usually 20,000 psi, although some specify 18,000 psi, together with smaller concrete stresses than those given above.

To find the resisting moment in terms of the stresses, moments can be taken first about the steel to get an expression for the stress in the concrete. The resultant compressive stress is at the center of gravity of the triangle *CFO* of Fig. 232. The resultant tensile stress is regarded as being at the center of the reinforcement; therefore, the arm of the resisting moment is $[1 - (k/3)]d$. The term $[1 - (k/3)]$ is represented by the single letter j.

$$\text{Resisting moment arm} = \left(1 - \frac{k}{3}\right) d = jd \qquad (159.1)$$

The resisting moment is the total force in the concrete multiplied by the moment arm.

$$M = \frac{f_c jkbd^2}{2} \qquad (159.2)$$

It is also the total force in the steel multiplied by its moment arm

$$M = f_s Ajd \qquad (159.3)$$

Problems

159-1. Concrete which tests 3,000 psi at 28 days is used for a beam 10 in. wide and 12 in. deep to the center of reinforcing. The reinforcement consists of three deformed bars, each having a cross section of 0.39 sq in. The beam weighs 125 lb per linear foot, is supported at the ends on a 15-ft span, and carries a concentrated load at the middle. If the allowable stress for the steel is 20,000 psi, find the resisting moment. Find the resisting moment based on the allowable stress in the concrete. Which governs? Find the concentrated load. *Ans.* $M = 238{,}000$ in.-lb; 375,000 in.-lb; $P = 4{,}400$ lb.

159-2. Solve the previous problem if the reinforcing consists of three ¾-in. square bars.

159-3. A 12- by 15-in. beam is supported on a 20-ft span. It weighs 200 lb per ft and carries two equal loads, each load being 5 ft from a support. The reinforcing consists of three ⅞-in. square bars. If $n = 10$, $f_c = 1{,}350$ psi, and $f_s = 20{,}000$ psi, find the magnitude of the two loads.

In designing beams to carry a given load, it is necessary to make some assumptions to get started. The value of j will usually be between 0.85 and 0.87, but if j is assumed ⅞ and k is ⅜, Eq. (159.2) may be written

$$bd^2 = \frac{2M}{(0.375)(0.875)f_c} = \frac{6M}{f_c} \text{ approximately}$$

from which the dimensions can be selected and the area of steel found from

$$A = \frac{M}{f_s(0.875)d}$$

It is, of course, necessary to check the design and make changes to improve the efficiency.

Problem

159-4. Design a reinforced-concrete beam for a span of 20 ft to carry a load of 800 lb per ft including its own weight.

160. Steel Ratio. The effect produced by changing the percentage of reinforcement is shown in Prob. 159-1 and 159-2. The objective of good design is to keep both concrete and steel stressed up to their allowable values. The ratio of the steel area to total area may be found for any allowable unit stresses. From the equality of the total tensile and compressive stress,

$$\frac{f_c kbd}{2} = f_s A$$

from which

$$k = \frac{2f_s A}{f_c bd} = \frac{2f_s p}{f_c}$$

From Eq. (158.8),

$$k^2 + 2pnk = 2pn$$

Eliminating k,

$$\frac{4f_s^2 p^2}{f_c^2} + \frac{4f_s p^2 n}{f_c} = 2pn$$

$$p = \frac{1}{2\dfrac{f_s}{f_c}\left(\dfrac{f_s}{nf_c} + 1\right)}$$

This gives the value of p for *balanced reinforcement.*

Problem

160-1. Find the steel ratio if the allowable unit compressive stress in the concrete is 1,350 psi, the allowable tensile stress in the steel is 20,000 psi, and the ratio of the modulus of elasticity of the steel to that of the concrete is 10.

161. Bond and Shearing Stresses. In order to make the steel reinforcing effective there must be good bond between the rods and the concrete. Let Σ_0 represent the perimeter of the rods and u the unit bond stress. In a length of beam Δx the total bond force must equal the increase in tension in the steel

$$u\Sigma_0\,\Delta x = \Delta f_s A \tag{161.1}$$

From Eq. (159.3),

$$\Delta M = \Delta f_s A jd$$

$$u = \frac{\Delta M}{\Delta x jd\Sigma_0} = \frac{V}{\Sigma_0 jd} \tag{161.2}$$

The shearing stress on any horizontal plane between the neutral axis and the steel rods may be computed by equating the total shearing force on a length of the beam Δx to the increase in total compressive force. Let v be the unit shearing stress in the concrete. Then

$$vb\,\Delta x = \tfrac{1}{2}\Delta f_c kbd \tag{161.3}$$

From Eq. (159.2),

$$\Delta M = \tfrac{1}{2}\Delta f_c jkbd^2$$

$$v = \frac{\Delta M}{\Delta x jbd} = \frac{V}{bjd} \tag{161.4}$$

Most reinforcing bars are deformed to grip the concrete. The Joint Committee recommends an allowable bond stress equal to $0.05f'_c$, where f'_c is the ultimate strength of a concrete cylinder at 28 days. Often the end of the bar is bent in the shape of a hook to provide anchorage. The allowable shearing stress in concrete recommended is $0.02f'_c$. When necessary to provide for shear, vertical stirrups are used, or the tension reinforcement is bent at 45° to take the diagonal tension which is caused by shear.

Problems

161-1. Find the maximum bond stress and shearing stress in the concrete for Prob. 159-1. *Ans.* 29 psi; 22 psi.

161-2. Find the maximum bond stress and maximum shearing stress in the concrete for the beam of Prob. 159-2. Tell where these stresses occur.

161-3. For the beam of Prob. 159-3, find the maximum bond and shearing stress in the concrete.

CHAPTER 17

ELASTIC ENERGY OF BENDING AND SHEAR

162. Energy of External Work. The work done by a force is the product of the average force multiplied by the displacement of its point of application in the direction of the force. For elastic bodies,

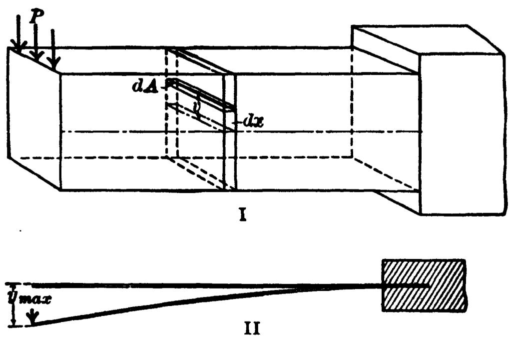


FIG. 233. Work of deflection.

the average force is one-half the sum of the initial and final forces. When a beam is deflected by a force P, the average force is $P/2$. For a cantilever with a load on the free end (Fig. 233),

$$y_{max} = \frac{Pl^3}{3EI}$$

$$\text{External work} = U = \frac{P}{2} \times \frac{Pl^3}{3EI} = \frac{P^2l^3}{6EI} \qquad (162.1)$$

For a simply supported beam which is loaded at the middle,

$$U = \frac{P}{2} \times \frac{Pl^3}{48EI} = \frac{P^2l^3}{96EI} \qquad (162.2)$$

When the elastic energy of a concentrated load is known, the deflection may be calculated from

$$\frac{Py}{2} = U \qquad y = \frac{2U}{P} \qquad (162.3)$$

Example

A simply supported beam of length l carries a load P at a distance a from the left end and a distance b from the right end. Find the deflection under the load (see Fig. 151).

The portion of length a is a cantilever which is pushed upward by the reaction Pb/l; the length b is another cantilever. From Eq. (162.1)

$$U = \frac{P^2b^2}{l^2} \times \frac{a^3}{6EI} + \frac{P^2a^2}{l^2} \times \frac{b^3}{6EI} = \frac{P^2b^2a^2}{6EIl^2}(a + b)$$

$$U = \frac{P^2b^2a^2}{6EIl} = \frac{Py}{2} \qquad y = \frac{Pa^2b^2}{3EIl}, \text{ positive downward}$$

Problems

162-1. How much work is done on a 3- by 4-in. cantilever 6 ft long when a load of 100 lb is placed on the free end if $E = 1{,}200{,}000$ psi? How much additional work is done when a second load of 100 lb is added?
Ans. $U = 32.4$ in.-lb; $U = 97.2$ in.-lb.

162-2. In Prob. 162-1, what was the average force when the first 100 lb was applied? What was the average force when the second 100 lb was applied?

162-3. A 6- by 8-in. beam 16 ft long is supported at 6 ft from the left end and held down at the right end. What is the total work when a load of 600 lb is placed on the left end if $E = 1{,}200{,}000$ psi? Calculate the work as two cantilevers. *Ans.* $U = 72.9 + 121.5 = 194.4$ in.-lb; $y = 0.648$ in.

When there is a uniformly distributed load of w pounds per unit length, the increment of load on a length dx is $w\,dx$, and the work done by this increment is $wy\,dx/2$, in which y is the deflection of the particular part of the beam upon which the increment rests. In Fig. 234,II,

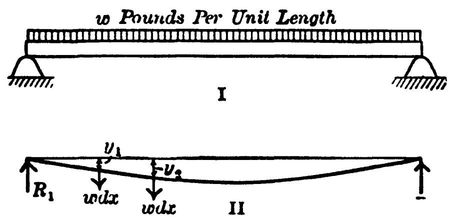


FIG. 234. External work.

one increment $w\,dx$ is deflected a distance y_1, a second increment is deflected a distance y_2, etc.

The different values of y are determined from the equation of the elastic line. The total work is the sum of these increments of work.

$$\text{Total work} = \frac{w}{2}\int y\,dx \qquad (162.4)$$

with the ends of the beam as the limits.

The equation of the elastic line for a simply supported uniformly loaded beam (from Art. 84) is

$$y = \frac{w}{24EI}(l^3x - 2lx^3 + x^4)$$

if y is taken positive downward.

$$dU = \frac{w^2}{48EI}(l^3x - 2lx^3 + x^4)\,dx \tag{162.5}$$

$$U = \frac{w^2}{48EI}\left[\frac{l^3x^2}{2} - \frac{lx^4}{2} + \frac{x^5}{5}\right]_0^l = \frac{w^2l^5}{240EI} \tag{162.6}$$

Problem

162-4. Find the total external work on a uniformly loaded cantilever.

Ans. $U = w^2l^5/40EI$.

163. Internal Work in a Beam. The unit stress at a distance v from the neutral axis of a beam is Mv/I. The internal work or resilience per unit volume is $s^2/2E$. Figure 233,I shows an element of volume of cross section dA and length dx at a distance v from the neutral surface. The volume of this element is $dA\ dx$ and the internal energy is

$$dU = \frac{s^2}{2E}\,dA\,dx = \frac{M^2v^2}{2EI^2}\,dA\,dx \tag{163.1}$$

$$\text{Total work in beam} = \int\int \frac{M^2}{2EI^2}\,v^2\,dA\,dx \tag{163.2}$$

The integration of this equation with respect to v as the variable gives the work done upon the volume of length dx between two vertical planes. Throughout this volume, x, M, and I are constant. The integral of $v^2\,dA$ across the beam from the bottom to the top is I.

$$\text{Work} = \int \frac{M^2}{2EI}\,dx \tag{163.3}$$

This equation is used to calculate the internal work in any beam. Unless M and I are constant, they must be expressed as functions of x before integrating.

For a uniform beam under constant moment M, Eq. (163.3) becomes

$$U = \frac{M^2}{2EI}\int_0^l dx = \frac{M^2}{2EI}\Big[x\Big]_0^l = \frac{M^2l}{2EI} \tag{163.4}$$

For a beam supported at the ends with uniformly distributed load, $M = (wlx/2) - (wx^2/2)$.

$$U = \frac{w^2}{8EI}\int_0^l (l^2x^2 - 2lx^3 + x^4)\,dx \tag{163.5}$$

$$U = \frac{w^2}{8EI}\left[\frac{l^2x^3}{3} - \frac{lx^4}{2} + \frac{x^5}{5}\right]_0^l = \frac{w^2l^5}{240EI} \tag{163.6}$$

which agrees with Eq. (162.6).

For a cantilever with a load on the free end, $M = -Px$. When the section is constant,

$$U = \frac{P^2}{2EI}\int_0^l x^2\,dx = \frac{P^2}{6EI}\Big[x^3\Big]_0^l = \frac{P^2l^3}{6EI} \tag{163.7}$$

For a beam supported at the ends with a load P at a distance a from one end and at a distance b from the other (Fig. 151), the reaction at the end of the length a is Pb/l and the moment in this length is Pbx/l. The work in this part of the beam is

$$U = \frac{P^2b^2}{2EIl^2}\int_0^a x^2\,dx = \frac{P^2b^2a^3}{6EIl^2} \tag{163.8}$$

Similarly, in the length b,

$$\text{Work} = \frac{P^2a^2b^3}{6EIl^2} \tag{163.9}$$

For the entire length,

$$\text{Total work} = \frac{P^2a^2b^2(a+b)}{6EIl^2} = \frac{P^2a^2b^2}{6EIl} \tag{163.10}$$

When the load is at the middle, $a = b = l/2$ and

$$\text{Total work} = \frac{P^2l^3}{96EI} \tag{163.11}$$

Problem

163-1. Find the total internal work in a uniformly loaded cantilever.

Ans. $U = w^2l^5/40EI$.

164. Energy in Unit Volume. Since the stress in any cross section of a beam increases from zero at the neutral axis to a maximum stress S in the outer fibers, the average energy per unit volume in any short portion is considerably smaller than $S^2/2E$. For all beams, except beams of constant strength or beams under constant moment, the stress in the outer fiber varies along the length and has its maximum value only at the dangerous section. For these reasons, the efficiency of a beam as a method of storing elastic energy depends on the form of the section and the arrangement of the loads. The most efficient beam is one in which the greatest portion of the material is brought to a relatively high stress.

For a beam of uniform section under constant moment,

$$U = \frac{M^2l}{2EI} \tag{164.1}$$

This equation gives the total work in terms of the bending moment. It is often desirable to find the work in terms of the unit stress in the extreme fibers. If the neutral axis is midway between the extreme top and bottom fibers, $c = d/2$, and $M = 2SI/d$. When this value of the moment is substituted in Eq. (164.1), the result is

$$U = \frac{4S^2I^2l}{2EId^2} = \frac{2S^2Il}{Ed^2} \tag{164.2}$$

For a rectangular section,

$$U = \frac{2S^2b\,d^3l}{12Ed^2} = \frac{S^2b\,d\,l}{6E} = \frac{S^2}{6E} \times \text{volume} \tag{164.3}$$

The average energy per unit of volume is $S^2/6E$, which is one-third as great as that in a block subjected to a uniform stress S.

Problems

164-1. Find the average energy per unit volume in a solid circular section subjected to a uniform bending moment. *Ans.* $S^2/8E$.

164-2. A steel bar 2 in. wide and ½ in. thick is 8 ft. long and rests on two supports 6 ft apart and carries two equal loads on the ends. If E is 30,000,000 psi, what is the total elastic energy in the part between the supports when each load on the ends is 100 lb? *Ans.* 82.9 in.-lb.

For a cantilever with unifomly distributed load,

$$U = \frac{w^2l^5}{40EI} = \frac{W^2l^3}{40EI} \tag{164.4}$$

For a section which is symmetrical with respect to the neutral axis

$$\frac{Wl}{2} = \frac{2SI}{d}$$

$$U = \frac{4S^2Il}{10Ed^2} \tag{164.5}$$

For a rectangular section, for which $I = bd^3/12$,

$$\text{Total work} = \frac{S^2b\,d\,l}{30E} = \frac{S^2}{30E} \times \text{volume} \tag{164.6}$$

The total energy in a cantilever of rectangular section with uniformly distributed load is one-fifteenth as much as that in a block of the same volume with uniform compressive stress throughout.

165. Internal Work in a Shaft. The unit shearing stress s_s produces a deformation of s_s/G in planes at unit distance apart. The work of shear is the product of half the unit stress by the unit deformation.

$$\text{Work per unit volume} = \frac{s_s}{2} \times \frac{s_s}{G} = \frac{s_s^2}{2G} \tag{165.1}$$

In a solid circular shaft at a distance r from the axis, the unit shearing stress is kr and

$$\text{Energy per unit volume} = \frac{k^2r^2}{2G} \tag{165.2}$$

The element of volume of length l is $2\pi rl\,dr$ and

$$\text{Total energy} = \frac{\pi k^2 l}{G}\int_0^a r^3\,dr = \frac{\pi l k^2 a^4}{4G} \tag{165.3}$$

in which a is the radius of the shaft. The maximum unit shearing stress in the outer surface is $S_s = ka$, and

$$\text{Total energy of shear} = \frac{S_s^2}{4G}\pi a^2 l = \frac{S_s^2}{4G} \times \text{volume} \tag{165.4}$$

Since the modulus of elasticity in shear is about two-fifths as great as the modulus in tension or compression, the total energy of a rod in torsion, for equal values of the unit stress, is one-fourth greater than that of the same rod in tension. However, since the elastic limit of steel and other similar materials in shear is somewhat smaller than in tension, the total energy which may be stored is approximately the same in both cases.

166. Work of Shear in a Rectangular Beam. In a beam of rectangular section of breadth b and depth d subjected to a vertical shear V,

$$s_s = \frac{V}{Ib}\int bv\,dv = \frac{V}{I}\left[\frac{v^2}{2}\right]_v^{d/2} = \frac{V}{8I}(d^2 - 4v^2) \tag{166.1}$$

$$\frac{s_s^2}{2G} = \frac{V^2}{128GI^2}(d^4 - 8d^2v^2 + 16v^4) \tag{166.2}$$

When the second member of Eq. (166.2) is multiplied by the element of volume, which is $b\,dv\,dx$, and the product is integrated with respect to v with the limits of $-d/2$ and $d/2$, the result is

$$U = \int \frac{V^2bd^5}{128GI^2}\left(1 - \frac{2}{3} + \frac{1}{5}\right)dx = \int \frac{3V^2}{5Gbd}\,dx \tag{166.3}$$

When V is constant, the last term of Eq. (166.3) for a beam of constant section is $U = \frac{3V^2l}{5Gbd}$. For a cantilever beam with a load on the free

end, $V = -P$ and

$$U = \frac{3P^2 l}{5Gbd} \tag{166.4}$$

To find the deflection which is caused by shear at the end of a cantilever with a load on the end,

$$\frac{Py_s}{2} = \frac{3P^2 l}{5Gbd}$$

$$y_s = \frac{6Pl}{5Gbd} \tag{166.5}$$

For a beam supported at the ends and carrying a concentrated load P at the middle,

$$y_s = \frac{3Pl}{10\,Gbd} \tag{166.6}$$

Another method of finding the deflection caused by shear consists of integrating Eq. (88.2). The result obtained does not, however, check the above formulas because in the analysis of shear energy it was assumed that all cross sections of the beam were free to warp. For a cantilever the sections near the wall are restrained and do not fulfill this condition. The same may be said of sections near the middle of a simply supported beam with a concentrated load at the center.[1]

Problems

166-1. A 2- by 3-in. steel cantilever is 30 in. long and carries a load of 1,800 lb on the free end. Find the deflection of bending and the deflection of shear if $E = 30{,}000{,}000$ psi and $G = 12{,}000{,}000$ psi.

Ans. $y_b = 0.12$ in.; $y_s = 0.0009$ in.

166-2. Solve Prob. 166-1 for a length of 15 in. and a load of 3,600 lb.

Ans. $y_b = 0.030$ in.; $y_s = 0.0009$ in.

167. Work of Two Loads. When the expression

$$U = \int \frac{M^2}{2EI}\,dx \qquad \text{Formula XXV}$$

applies for two or more loads, the moments for the separate loads must be added together and then squared to give M^2 for integration. For a cantilever with a uniform load of w per unit length and a concentrated load P on the free end, the moment is $-Px - (wx^2/2)$.

[1] For a good discussion of this topic, see S. Timoshenko, "Strength of Materials," 2d ed., Part I, pp. 296–300, D. Van Nostrand Company, Inc., New York, 1940.

$$U = \frac{1}{2EI}\int_0^l \left(P^2x^2 + Pwx^3 + \frac{w^2x^4}{4}\right)dx =$$

$$\frac{1}{2EI}\left[\frac{P^2x^3}{3} + \frac{Pwx^4}{4} + \frac{w^2x^5}{20}\right]_0^l \quad (167.1)$$

$$U = \frac{P^2l^3}{6EI} + \frac{Pwl^4}{8EI} + \frac{w^2l^5}{40EI} \quad (167.2)$$

The first term of Eq. (167.2) after the equality sign is the internal work which is done by the load P alone, as shown by Eq. (162.1). The last term is the work which is done by the distributed load alone, as shown by Prob. 162-4. The intermediate term, which includes both P and w, must be the two loads together. If the load P is placed first on the beam, the average force during its own deflection is $P/2$ and the work is

$$U = \frac{P}{2} \times \frac{Pl^3}{3EI} = \frac{P^2l^3}{6EI}$$

When the distributed load wl $(= W)$ is placed on the beam, the *additional* deflection at the *end* is $wl^4/8EI(= Wl^3/8EI)$. Since the full load P is on the end of the beam while this deflection takes place, the work done by P is

$$U = P \times \frac{wl^4}{8EI} = \frac{Pwl^4}{8EI} = \frac{PWl^3}{8EI}$$

Problems

167-1. A cantilever beam 9 ft long is 4 by 6 in. and has $E = 1{,}500{,}000$ psi. It carries 25 lb per ft uniformly distributed and 200 lb concentrated on the free end. Using Formula XXV; find the total energy in the beam.
Ans. $U = 158.13$ in.-lb.

167-2. Solve Prob. 167-1 if the uniform load extends over the outside 4 ft of the beam with no load on the interior 5 ft. The 200-lb load remains on the free end.

167-3. A 6- by 1-in. rectangular beam 9 ft long rests on supports which are 2 ft from the ends. The beam is weightless and $E = 1{,}200{,}000$ psi. Find the total internal energy caused by placing a 30-lb concentrated load on each free end. By equating the internal energy to the external energy, find the deflection of the ends of the beam. *Ans.* $y = 1.0944$ in.

168. Maxwell's Theorem of Reciprocal Deflections. Figure 235 represents a beam which rests at the ends on fixed supports or on horizontal planes so that no work is done by displacement at these supports. Figure 235,II shows the elastic line of this beam when a load P is applied at the point A. The deflection of A from its original position is y_p. If y_a represents the deflection at A which would be caused by unit

load at A, the deflection caused by the load P is Py_a; hence

$$y_p = Py_a \tag{168.1}$$

The load P at A causes a deflection at another point B, which may be called y_{pb}. If y_{ab} is the deflection at B which would result from a unit load at A, then the deflection caused by a load P at A would be Py_{ab}, and

$$y_{pb} = Py_{ab} \tag{168.2}$$

Figure 235,III shows the elastic line of this beam with a load Q at B and no load at A. The deflection at B is

$$y_q = Qy_b \tag{168.3}$$

in which y_b is the deflection at B which would be caused by unit load at B. The deflection at A is

$$y_{qa} = Qy_{ba} \tag{168.4}$$

in which y_{ba} is the deflection at A which would be caused by unit load at B.

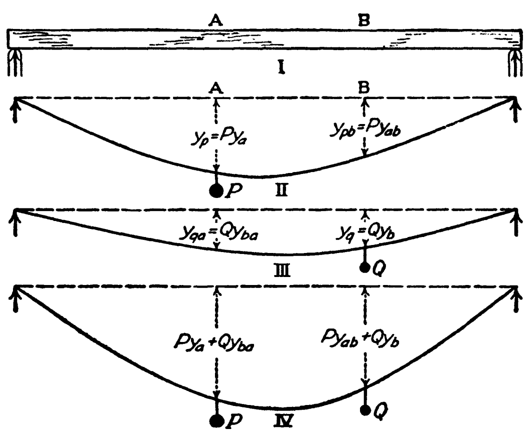


FIG. 235. Reciprocal deflections.

Figure 235,IV shows the elastic line for the beam when both P and Q are applied. If the load P is applied first, the average $P/2$ multiplied by y_p gives the work at A.

$$U_{pa} = \frac{Py_p}{2} = \frac{P^2y_a}{2} \tag{168.5}$$

Since there is no force at B, the deflection $y_{pb} = Py_{ab}$ represents no external work.

When the load Q is applied at B, the additional deflection at A is $y_{qa} = Qy_{ba}$, which is the same as it would be if there were no load at A. Since the load P at A moves the distance y_{qa}, which equals Qy_{ba}, the additional work at A is

$$U_{qa} = PQy_{ba} \tag{168.6}$$

At B, when the load Q is applied, the work is

$$U_{qb} = \frac{Qy_q}{2} = \frac{Q^2y_b}{2} \tag{168.7}$$

$$\text{Total } U = U_{pa} + U_{qa} + U_{qb} = \frac{P^2y_a}{2} + PQy_{ba} + \frac{Q^2y_b}{2} \tag{168.8}$$

When the load Q is applied first, the first and last terms of the energy expression are the same as in Eq. (168.8). When the load P is applied, B moves a distance Py_{ab} and takes the load Q that distance. The additional work at B is then

$$U_{pb} = QPy_{ab} \tag{168.9}$$

$$\text{Total } U = U_{pa} + U_{pb} + U_{qb} = \frac{P^2y_a}{2} + QPy_{ab} + \frac{Q^2y_b}{2} \tag{168.10}$$

Equation (168.8) gives the total energy when P is applied before Q; Eq. (168.10) gives the total work when Q is applied before P. By the principle of *superposition* which is proved by experiment and theory, the deflection caused by several loads is the sum of the deflections caused by the loads acting separately (provided, of course, that the proportional limit is not exceeded); and the order of application of the loads is immaterial. Since Eqs. (168.8) and (168.10) represent the same total energy,

$$QPy_{ab} = PQy_{ba} \qquad y_{ba} = y_{ab} \tag{168.11}$$

Equation (168.11) is *Maxwell's theorem of reciprocal deflection.* The deflection at a point A caused by a load at B is equal to the deflection at B caused by the same load at A.

There is nothing in the derivation of the equations which limits Maxwell's theorem to a simply supported beam. It applies to any determinate or indeterminate frame.

169. Castigliano's Theorem. If Eq. (168.8) or (168.10) is differentiated with respect to the load P, the result is

$$\frac{\partial U}{\partial P} = Py_a + Qy_{ab} \tag{169.1}$$

of which the first term Py_a is the deflection at A caused by the load P, and the second term Qy_{ba} is the deflection at A caused by the load Q. Castigliano's theorem states that the *derivative of the total elastic energy with respect to any concentrated load gives the total deflection at that load.* To get the deflection at any point at which there is no concentrated load, a "dummy" load P is assumed at that point. The last term of Eq. (169.1) then gives the total deflection. If R is the reaction at a support of an indeterminate beam, $\partial U/\partial R$ is the deflection at that point. Since this deflection is zero at a fixed support, $\partial U/\partial R = 0$. This is the so-called method of "least work," which is really no work.

Example

Find the equation of the elastic line for a uniformly loaded cantilever.

If a dummy load P is placed at any distance a from the free end, the moment equation is

$$M = -\frac{wx^2}{2} - P(x - a) \tag{169.2}$$

$$M^2 = \frac{w^2x^4}{4} + Pw(x^3 - ax^2) + P^2(x - a)^2 \tag{169.3}$$

Since the first term of Eq. (169.3), $w^2x^4/4$, does not contain P, the derivative of the energy expression from this term with respect to P as the variable will be zero; hence this term need not be integrated. The last term $P^2(x - a)^2$ gives the deflection caused by P alone. Since P is a dummy force and the deflection caused by the distributed load is required, this term is not integrated. (If P were an actual load and the total deflection were desired under this load, then this last term would be integrated.)

$$U = \frac{Pw}{2EI}\int_a^b (x^3 - ax^2)\,dx = \frac{Pw}{2EI}\left[\frac{x^4}{4} - \frac{ax^3}{3}\right]_a^l \tag{169.4}$$

$$U = \frac{Pw}{2EI}\left(\frac{l^4}{4} - \frac{al^3}{3} + \frac{a^4}{12}\right) = \frac{Pw}{24EI}(3l^4 - 4al^3 + a^4)$$

$$y = \frac{\partial U}{\partial P} = \frac{w}{24EI}(3l^4 - 4al^3 + a^4), \text{ positive downward} \tag{169.5}$$

The term which was integrated is the product of $(wx^2/2) \times P(x - a)$, of which $wx^2/2$ extends from 0 to l; $P(x - a)$ extends from a to l. The integral of the product has the limits a and l.

170. Elastic-energy Method. If M_q represents the moment of any number of applied forces or couples and M_p represents the moment of a single concentrated force,

$$M^2 = (M_p + M_q)^2 = M_q^2 + 2M_pM_q + M_q^2 \tag{170.1}$$

$$U = \int \frac{M_p^2\,dx}{2EI} + \int \frac{M_p\,M_q\,dx}{EI} + \int \frac{M_q^2\,dx}{2EI} \tag{170.2}$$

in which each term is equivalent to the corresponding term of Eq. (168.8) or (168.10). In the elastic-energy method, the second term of Eq. (170.2) is used exclusively. The force P is a dummy or auxiliary

load. For a single concentrated load, which could be solved conveniently by the last term of Eq. (170.2), it is customary to use the second term with the dummy force to avoid confusion. While the final equations are practically the same as those obtained by Castigliano's methods, the derivation of the method is simpler and graphic integration is more convenient.[1]

$$\int \frac{M_p M_q\,dx}{EI} = Py_{qa} \tag{170.3}$$

which is the work done by the force P when its point application at A is displaced a distance y_{qa} by forces which produce the moments M_q. From Eq. (170.3), the deflection at A is

$$y = y_{qa} = \frac{1}{P}\int \frac{M_p\,M_q\,dx}{EI} \tag{170.4}$$

For uniform beams,

$$EIy = \frac{\int M_p\,M_q\,dx}{P} \tag{170.5}$$

If the force P is unity and the beam is uniform,

$$EIy = \int M_p\,M_q\,dx \tag{170.6}$$

Example

Derive the equation of the elastic line for a simply supported, uniformly loaded beam.

In Fig. 236, the auxiliary force is upward at A at a distance a from the left support and a distance b from the right support. The reaction downward at the

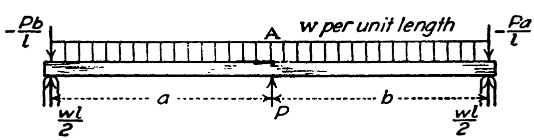


FIG. 236. Auxiliary or dummy force.

left end is $-Pb/l$, and at the right end is $-Pa/l$. Integration from each end to A gives less complicated expressions.

$$EIy = \frac{1}{P}\int_0^a \left(\frac{wlx}{2} - \frac{wx^2}{2}\right)\left(-\frac{Pbx}{l}\right)dx + \frac{1}{P}\int_0^b \left(\frac{wlx}{2} - \frac{wx^2}{2}\right)\left(-\frac{Pax}{l}\right)dx \tag{170.7}$$

[1] This method is largely used in continental Europe. For a full discussion, which includes trusses as well as beams, see J. A. Van den Broek, "Elastic Energy Theory," 2d ed., John Wiley & Sons, Inc., New York, 1942.

$$EIy = -\frac{wb}{l}\left[\frac{lx^3}{6} - \frac{x^4}{8}\right]_0^a - \frac{wa}{l}\left[\frac{lx^3}{6} - \frac{x^4}{8}\right]_0^b \tag{170.8}$$

$$EIy = -\frac{wab}{6}(a^2 + b^2) + \frac{wab}{8l}(a^3 + b^3) \tag{170.9}$$

$$EIy = -\frac{wab}{24}(a^2 + 3ab + b^2) \tag{170.10}$$

Problems

170-1. Derive the equation of the elastic line for a uniformly loaded cantilever.
170-2. Substitute $a = x$, $b = l - x$, in Eq. (170.10) and compare with Eq. (84.7).

171. Graphic Integration. The example of the preceding article shows that the method of elastic energy by algebraic integration offers no advantage over previous methods for deriving the equation of the elastic line of straight beams. Graphic integration, however, may be made less complicated.

If the M_p diagram is bounded by straight lines and if the area of the M_q diagram (of any form whatever) is A_q, then

$$\int M_p M_q \, dx = A_q h_p \tag{171.1}$$

in which h_p is the altitude of the M_p diagram under the center of gravity of the M_q diagram.[1]

Since M_p is the moment of a concentrated load and reactions, its equation is of the first degree and may be represented by $M_p = a + bx$.

$$\int M_p M_q \, dx = \int (a + bx) M_q \, dx = a \int M_q \, dx + b \int x M_q \, dx$$
$$= aA_q + b\bar{x}A_q$$

$$\int M_p M_q \, dx = (a + b\bar{x}) A_q \tag{171.2}$$

in which $\bar{x}$ is the abscissa of the center of gravity of A_q, and $a + b\bar{x}$ is the ordinate of the M_p diagram which is under the center of gravity of the M_q diagram.

Another improvement (possibly new) consists in reversing the M_p diagram, as shown in Fig. 237. Since the resultant moment from any set of forces is the same, no matter which way it is calculated, the result is the same whether the M_p and M_q diagrams go in the same or in opposite directions. If the dummy force is at a distance x from the left end, the dummy reaction at the right end is $-Px/l$ and the altitude of the negative dummy triangle is a simple function of x. Moreover, the areas OAB and OBC which accompany the positive dummy triangle

[1] GEDO, J. D., "Kräfteplan-Verfahren," p. 5, Alfred Kroener, Leipzig, 1932.

are a triangle and a parabola, instead of a trapezoid and a truncated parabola.

Example 1

Derive the equation of the elastic line for a simply supported, uniformly loaded beam.

Figure 237,I shows the uniformly loaded beam, and Fig. 237,II gives the M_q diagram. The dummy force P upward at a distance x from the left support causes

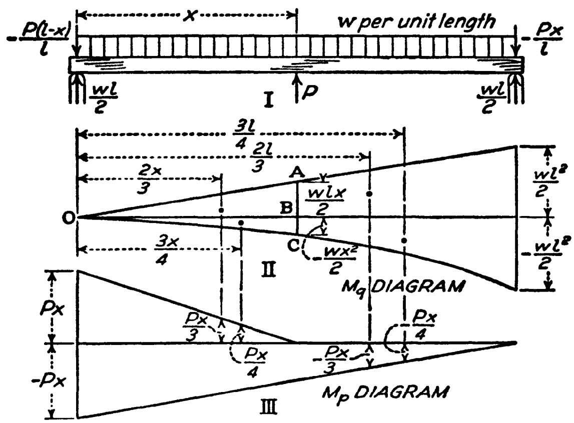


FIG. 237. Moment diagrams for load and for dummy force.

a downward reaction of $-Px/l$ at the right support. The M_p diagram, starting from the right end, consists of a negative triangle of base l and altitude $-Px$ and a positive triangle of base x and altitude Px.

$$\int M_p M_q \, dx = \frac{wl^3}{4} \times \left(-\frac{Px}{3}\right) - \frac{wl^3}{6} \times \left(-\frac{Px}{4}\right) + \frac{wlx^2}{4} \times \frac{Px}{3} - \frac{wx^3}{6} \times \frac{Px}{4} \tag{171.3}$$

$$\int M_p M_q \, dx = -\frac{Pwl^3x}{24} + \frac{Pwlx^3}{12} - \frac{Pwx^4}{24} \tag{171.4}$$

$$EIy = \frac{1}{P} \int M_p M_q \, dx = -\frac{wx}{24}(l^3 - 2lx^2 + x^3) \tag{171.5}$$

Example 2

Derive the equation of the elastic line for a simply supported beam which carries a load Q at a distance b from the right support. By using Fig. 238 with $P = 1$,

$$EIy = \frac{Qbl}{2} \times \left(-\frac{x}{3}\right) - \frac{Qb^2}{2} \times \left(-\frac{bx}{3l}\right) + \frac{Qbx^2}{2l} \times \frac{x}{3} \tag{171.6}$$

$$EIy = -\frac{Qblx}{6} + \frac{Qb^3x}{6l} + \frac{Qbx^3}{6l} = -\frac{Qbx}{6l}(l^2 - x^2 - b^2) \tag{171.7}$$

If the load Q is to the left of P, replace b in Eq. (171.7) by a and measure x from the right end.

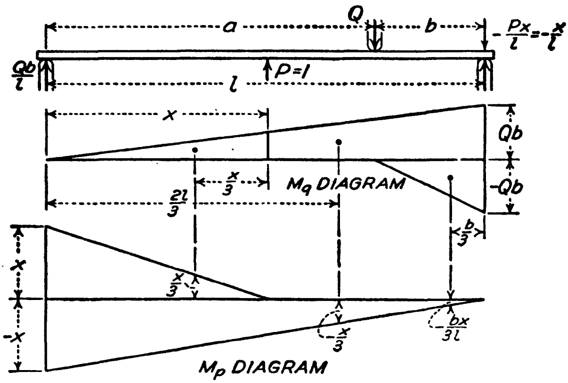


FIG. 238. Concentrated load and dummy diagrams.

Problems

171-1. Construct Fig. 238 with P on the right of Q and derive the expression in terms of x and a which must be added to Eq. (171.7) to get the deflection. Compare Prob. 97-1. *Ans.* $-Q(x - a)^3/6$.

171-2. A simply supported beam carries a uniformly distributed load over $0.6l$ adjacent to the right end. Derive the equation of the elastic line for the portion which is not loaded. Derive the equation of the elastic line for the loaded portion. *Ans.* $EIy = -0.0246wl^3x + 0.03wlx^3$.

172. Work of a Couple. When a couple of moment M_t, in a plane perpendicular to the neutral axis, is applied to any section of a beam, the work done by the couple is the moment multiplied by the angular rotation in radians.

$$U = M_t\,\theta \tag{172.1}$$

When the moment varies with the displacement, the work from zero displacement to θ is

$$U = \frac{M_t\,\theta}{2} \tag{172.2}$$

When M_q and M_t act on the same beam, Eq. (168.8) becomes

$$U = \int \frac{M_t^2\,dx}{2EI} + \int \frac{M_t\,M_q\,dx}{EI} + \int \frac{M_q^2\,dx}{2EI} \tag{172.3}$$

By using the second term of Eq. (172.3) with Eq. (172.1), since M_t is constant while the moment M_q is applied,

$$M_t\theta = \frac{1}{EI}\int M_t\,M_q\,dx \tag{172.4}$$

$$EI\theta = \int \frac{M_t\,M_q\,dx}{M_t} \tag{172.5}$$

If the magnitude of M_t is unity, Eq. (172.5) becomes

$$EI\theta = \int M_t M_q \, dx \tag{172.6}$$

The last three equations apply to beams of constant section only.

Example 1

Find the slope at the end and at any section for a cantilever with a load on the free end (Fig. 239).

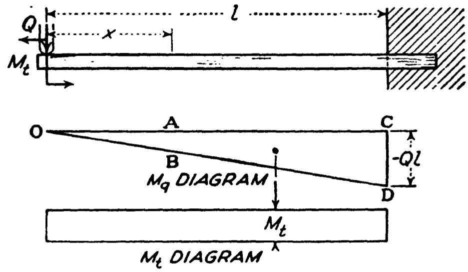


FIG. 239. Beam with auxiliary moment at end.

The couple M_t at the free end is transmitted as a constant couple to the fixed end.

$$EI\theta = \int_a^l \frac{M_t Qx \, dx}{M_t} = -\left[\frac{M_t Qx^2}{2M_t}\right]_a^l = -\frac{Q(l^2 - a^2)}{2} \tag{172.7}$$

By graphic integration, the slope at the end is

$$EI\theta = -\frac{Ql^2}{2} \times \frac{M_t}{M_t} = -\frac{Ql^2}{2} \tag{172.8}$$

For the slope at x, the trapezoid area is multiplied by M_t/M_t.

Problem

172-1. Find the slope of a uniformly loaded cantilever at the free end and at a distance x from the free end. Use Eq. (172.6) with algebraic and graphical integration.

Example 2

Find the slope at any point on a uniformly loaded beam (Fig. 240). The dummy moment M_t at a distance x from the left end is transmitted to both ends and balanced by a downward force M_t/l at the right support and an equal upward force at the left support, which together form a couple. With M_t equal to unity, the combined M_t diagram is $DCBA$. It is best to make the negative triangle of altitude -1 at the left end and the positive rectangle of altitude unity.

$$EI\theta = \frac{wl^3}{4} \times \left(-\frac{1}{3}\right) - \frac{wl^3}{6} \times \left(-\frac{1}{4}\right) + \frac{wlx^2}{4} \times 1 - \frac{wx^3}{6} \times 1 \tag{172.9}$$

$$EI\theta = -\frac{wl^3}{24} + \frac{wlx^2}{4} - \frac{wx^3}{6} \tag{172.10}$$

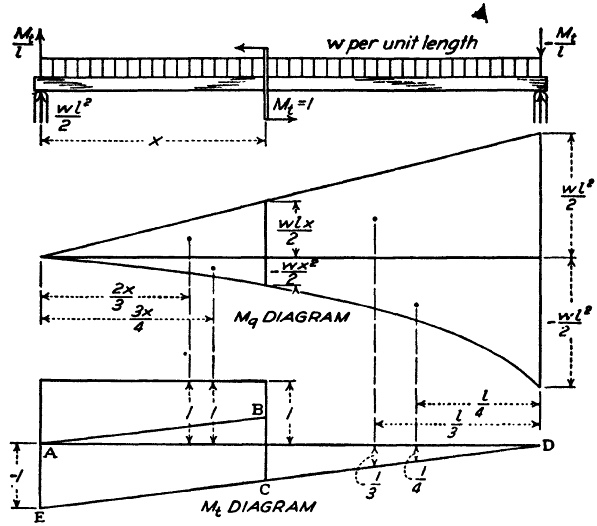


FIG. 240. Auxiliary moment at any section.

Problems

172-2. Construct the dummy triangle to find the slope at the left support of Fig. 240. Calculate the slope and check with Eq. (172.10).

172-3. Find the slope at the left end of a simply supported beam which carries a load Q at a distance b from the right support. Use the M_q diagram of Fig. 240, and draw an M_t diagram.

172-4. Solve Prob. 172-3 for the slope at any point to the left of the load. Use the M_t diagram of Fig. 240.

172-5. Solve Prob. 172-3 to find the slope to the right of the load.

172-6. A simply supported beam carries a uniformly distributed load over a length of $0.6l$ adjacent to the right end. Find the slope at the left end, at $0.4l$ from the left end, and at the middle.

Ans. $EI\theta = -0.0246wl^3$; $-0.0102wl^3$; $-0.0019wl^3$.

172-7. Find the deflection of the beam of Prob. 172-6 for any value of x which is not greater than $0.4l$. From this derive an expression for the slope and check the first two answers of Prob. 172-6.

173. Overhanging Beams. Figure 241 shows a beam which has a span of length l and overhangs the left support a distance a. When a load Q is placed on the left end, the upward reaction at the right support is reduced Qa/l. The beam may be regarded as made up of two cantilevers and the total work may be calculated by Eq. (162.1) without the use of a dummy force. Since all the external work is done by the load Q,

$$\frac{Q}{2}y_{max} = \frac{Q^2a^3}{6EI} + \frac{Q^2a^2l}{6EI} \tag{173.1}$$

$$y_{max} = \frac{Qa^3}{3EI} + \frac{Qa^2l}{3EI} \tag{173.2}$$

positive downward. The first term of the second member of Eq. (173.2) is recognized as the deflection of a cantilever from the tangent at the fixed end.

If θ_1 is the change of slope at the left support when the load Q is applied, the work of bending over the support is equal to the internal work in the span of length l,

$$\frac{Qa}{2}\theta_1 = \frac{Q^2a^2l}{6EI} \tag{173.3}$$

$$\theta_1 = \frac{Qal}{3EI} \tag{173.4}$$

The change of slope at the left support multiplied by the length a should give the last term of Eq. (173.2).

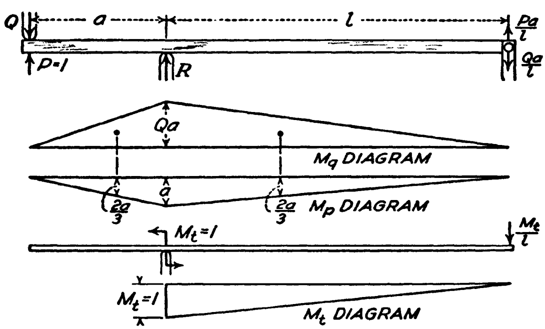


FIG. 241. Slope over support.

Problems

173-1. Find the deflection at the left end of the beam of Fig. 241 by the elastic-energy method, using the M_q and M_p diagrams.

173-2. Find the slope over the support of the beam of Fig. 241, using the M_q and M_t diagrams.

To find the deflection between the supports at a distance x from the left support, the M_p diagram is *usually* made for a simply supported beam. The M_q diagram is not changed but is drawn for the single span alone, since the product of $M_p\,M_q$ has no value unless both terms are included. Figure 242 shows these diagrams for the beam of Fig. 241. The M_p diagram is exactly like that of Fig. 237. With the M_q diagram as shown in Fig. 241, it would be convenient to reverse the M_p diagram and take the distance from the right support as x. In Fig. 242, the M_q diagram is drawn for two terms as the sum of a positive rectangle and a negative triangle. From this diagram,

$$EIy = Qal \times \left(-\frac{x}{2}\right) + \frac{Qal}{2} \times \left(-\frac{x}{3}\right) + Qax \times \frac{x}{2} - \frac{Qax^2}{2l} \times \frac{x}{3} \tag{173.5}$$

$$EIy = -\frac{Qalx}{3} + \frac{Qax^2}{2} - \frac{Qax^3}{6l} \tag{173.6}$$

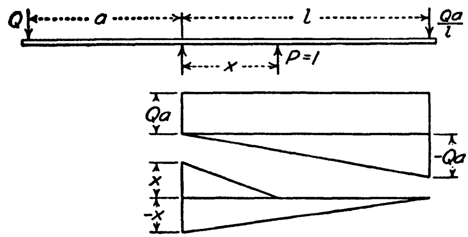


FIG. 242. Diagram for a single span.

Problems

173-3. Knowing the slope at the left support from Eq. (173.4), derive Eq. (173.6) by area moments or successive integration.

173-4. Reverse the M_p diagram of Fig. 242 with x from the right support and find the equation of the elastic line for the span of Fig. 241. Check with Eq. (173.6).

174. Span Fixed at One End. Figure 243 illustrates the indeterminate problem of a beam which is fixed at the right end, supported at a

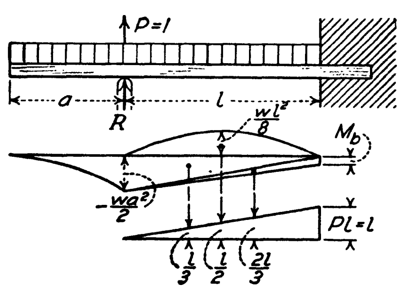


FIG. 243. Fixed-end span as a cantilever for dummy force.

distance a from the left end, and uniformly loaded. Either the moment at the fixed end or the reaction of the support may be taken as the unknown. The dummy force is applied at the left support with the beam regarded as a cantilever. This gives an M_p triangle of altitude l when P is taken as unity. Since the deflection at the support is zero,

$$EIy = 0 = -\frac{wa^2l}{4} \times \frac{l}{3} + \frac{M_bl}{2} \times \frac{2l}{3} + \frac{wl^2}{12} \times \frac{l}{2} \tag{174.1}$$

$$-\frac{wa^2l^2}{12}+\frac{M_bl^2}{3}+\frac{wl^4}{24}=0 \tag{174.2}$$

$$M_b=-\frac{wl^2}{8}+\frac{wa^2}{4} \tag{174.3}$$

Instead of the dummy force P at the left support a dummy moment M_t might be taken at the fixed end. The M_t diagram is a triangle like the M_p diagram and the calculation is the same.

Problems

174-1. Using Eq. (174.3), find the reaction at the support.

174-2. Write the moment diagram for the beam of Fig. 243 from the general moment equation $M_0 + V_0x - (wx^2/2)$. Draw the M_q diagram to represent each of these three terms and solve for the unknown shear.

174-3. A beam of length l is fixed at the right end, supported at the left end, and subjected to a load Q at a distance b from the right end and a distance a from the left end. Find the moment at the fixed end. Calculate the reaction of the support. Compare with Art. 107.

175. Two Spans, One End Fixed. Figure 244 shows a uniformly loaded beam 22 feet long, with the right end fixed, which is supported

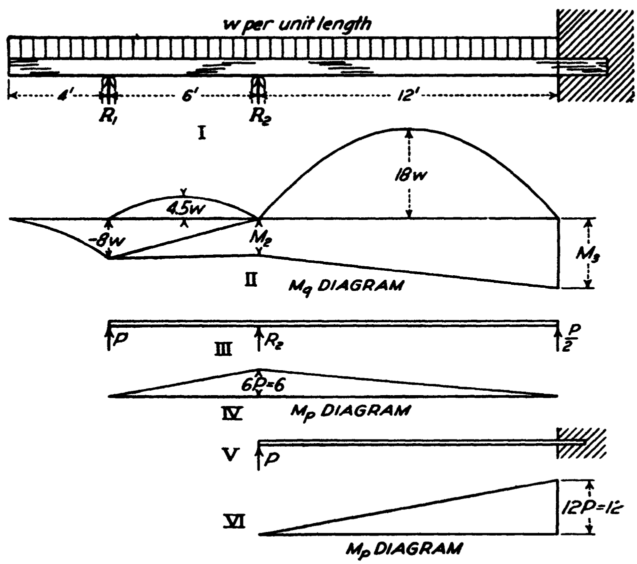


FIG. 244. Dummy diagrams for two spans and for one span.

4 feet from the left end and 10 feet from the left end. The moment over the second support and the moment at the fixed end are unknown. Figure 244,III shows the beam resting on the second support and at the right end. The dummy force P at the location of the left support

causes a force of $P/2$ at the right end. The dummy moment over the second support is $6P$, which equals 6 if P is unity.

$$0 = -24w \times 2 + 3M_2 \times 4 + 18w \times 3 + 6M_2 \times 4 + 6M_3 \times 2 + 144w \times 3 \quad (175.1)$$

$$36M_2 + 12M_3 + 438w = 0 \quad (175.2)$$

$$6M_2 + 2M_3 + 73w = 0 \quad (175.3)$$

The 12-foot span is used alone for the second equation. For the dummy moment, the span is assumed to be fixed at the right end. The force P at the location of the second support (Fig. 244,V) makes the triangular diagram of Fig. 244,VI. The span might be regarded as simply supported at the ends with a dummy moment M_1 at the fixed end, where the moment does no work. The final result is the same in each case.

$$0 = 6M_2 \times 4 + 6M_3 \times 8 + 144w \times 6 \quad (175.4)$$

$$M_2 + 2M_3 + 36w = 0 \quad (175.5)$$

$$M_2 = -7.4w \qquad M_3 = -14.3w$$

To derive the equation of the elastic line for any span of an indeterminate beam *after* the unknown moments and reactions have been calculated, the dummy diagram is drawn for a span of a simply supported beam. If either end is fixed, the span may be treated as a simply supported beam or a cantilever. The theory of Art. 168 was derived for a span which is fully supported, but not necessarily indeterminate, without the dummy force; and the applications of the principles of this article must conform to these restrictions.

In Fig. 244, the M_q diagrams are made of the end-moment diagram and the simple-support combined diagram. These are most convenient when dealing with end reactions and slope at a support. When working inside the span to find the elastic line or the slope at any point, it is desirable to draw the terms in such a way that no trapezoids or truncated parabolas will come next to the origin of coordinates. For the first span of Fig. 244, the general moment equation gives

$$-8w + 6V_0 - 18w = -7.4w \qquad V_0 = 3.1w$$

$$M_q = -8w + 3.1wx - \frac{wx^2}{2} \quad (175.6)$$

Example 1

Find the slope of the tangent at the left support of Fig. 244. The M_t diagram is a triangle of altitude 1 if M_1 is unity. Using the M_q formula,

$$EI\theta_1 = -48w \times (-\tfrac{1}{2}) + 55.8w \times (-\tfrac{1}{3}) - 36w \times (-\tfrac{1}{4}) \quad (175.7)$$

$$EI\theta_1 = 24w - 18.6w + 9w = 14.4w$$

Problems

175-1. Solve Example 1 using the M_q diagram of Fig. 244.

175-2. Derive the equation of the elastic line for the 6-ft span of Fig. 244, using the slope at the left support and the area-moment method with Fig. 245.

$$\textit{Ans.}\ EIy = 14.4wx - 4wx^2 + \frac{3.1wx^3}{6} - \frac{wx^4}{24} \tag{175.8}$$

Example 2

Derive the equation of the elastic line for the 6-ft span of Fig. 244 by elastic energy without using the slope at the end.

From the M_q and M_p diagrams of Fig. 245,

$$EIy = -48w \times \left(-\frac{x}{2}\right) + 55.8w \times \left(-\frac{x}{3}\right) - 36w \times \left(-\frac{x}{4}\right)$$
$$-8wx \times \frac{x}{2} + \frac{3.1wx^2}{2} \times \frac{x}{3} - \frac{wx^3}{6} \times \frac{x}{4} \tag{175.9}$$

Example 3

Find the slope at the left end of the 12-ft span of Fig. 245. Regard the dummy beam as a cantilever fixed at the right end which makes the M_t a rectangle of

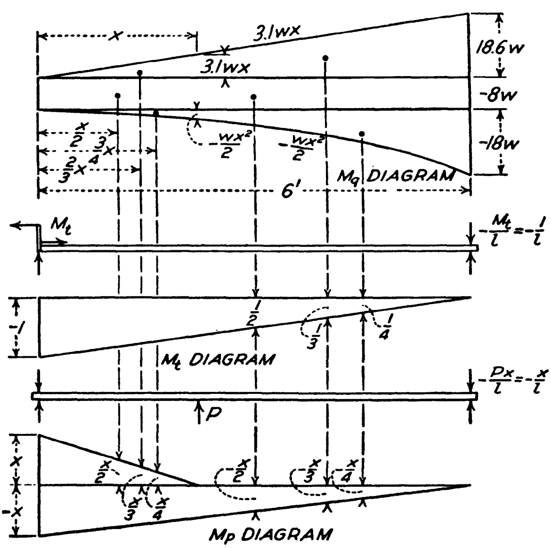


FIG. 245. Diagrams for slope and for deflection.

uniform height $-M_t = -1$. Use the M_q diagram of Fig. 244 and check with that of Fig. 246. Check also by means of the results of Example 1 and the area of the intervening moment diagram.

$$EI\theta_2 = -(7.4w + 14.3w)6 \times (-1) + 18w \times 8 \times (-1) = -13.8w$$

Example 4

Find the slope at the left end of the 12-ft span of Fig. 244 by means of the M_t diagram of Fig. 245 for a simply supported span. First derive the M_q equation from the data of preceding figures or problems.

Example 5

Derive the equation of the elastic line for the 12-ft span of Fig. 244 using the dummy moment line of Fig. 246 for fixed end.

On Fig. 246, the positive M_q trapezoid is taken as two triangles. The expression for the rectangle of length $12 - x$ is easily determined. The entire parabola of length 12 is taken and the value for the parabola of length x is subtracted.

$$EIy = -7.4w \times (12 - x) \times \frac{12 - x}{2} + 65.1w \times \frac{12 - x}{2} \times \frac{2(12 - x)}{3} +$$

$$\frac{21.7wx}{4} \times \frac{12 - x}{2} \times \frac{12 - x}{3} - 288w \times (9 - x) - \left(-\frac{wx^3}{6}\right) \times \left(-\frac{x}{4}\right) \qquad (175.10)$$

$$EIy = 18w(12 - x)^2 + \frac{21.7wx}{24}(12 - x)^2 - 288w(9 - x) - \frac{wx^4}{24} \qquad (175.11)$$

$$EIy = -13.8wx - 3.7wx^2 + \frac{21.7wx^3}{24} - \frac{wx^4}{24} \qquad (175.12)$$

Problems

175-3. Check Eq. (175.12) by substituting $x = 12$.

175-4. Solve Example 5 by area moments (or integration between limits), using the slope from Example 3 and the areas of length x of Fig. 246. Example 5

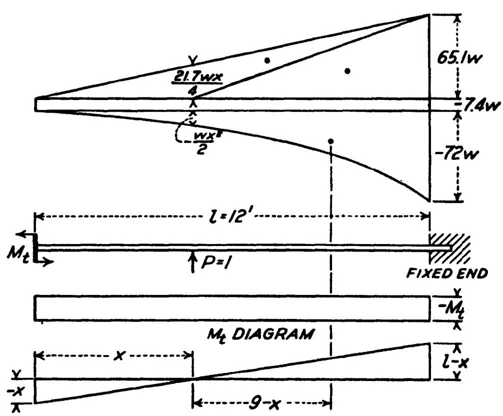


FIG. 246. Diagrams for slope and deflection.

is exactly the same as the deflection from the tangent at the right end by area moments.

175-5. Solve Example 5, using the reversed M_p diagram for the span as a simply supported beam, similar to that of Fig. 245. How does the difficulty of this method compare with that of Example 5?

176. Both Ends Fixed. Figure 247 shows a beam which is fixed at both ends, with a load Q at a distance a from the left end and b from the right end. The M_q diagram at the bottom is drawn with the simple-support portions separate. An M_t diagram is drawn with the right end fixed and the dummy moment at the left end. This is a rectangle. An M_p diagram is drawn with the left end fixed. This is a positive triangle. An equivalent triangle of different altitude would have been obtained if the beam were assumed to be simply supported

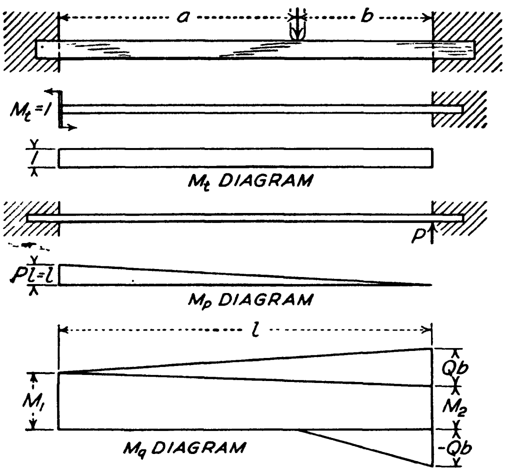


FIG. 247. Diagrams for beam fixed at both ends.

with a dummy moment at the left end. From the rectangular $M_t = -1$,

$$\left(\frac{M_1 l}{2} + \frac{M_2 l}{2} + \frac{Qbl}{2} - \frac{Qb^2}{2}\right) \times (-1) = 0 \tag{176.1}$$

$$M_1 l + M_2 l + Qbl - Qb^2 = 0 \tag{176.2}$$

From the triangular M_p,

$$\frac{M_1 l}{2} \times \frac{2l}{3} + \frac{M_2 l}{2} \times \frac{l}{3} + \frac{Qbl}{2} \times \frac{l}{3} - \frac{Qb^2}{2} \times \frac{b}{3} = 0 \tag{176.3}$$

$$2M_1 l + M_2 l + Qbl - \frac{Qb^3}{l} = 0 \tag{176.4}$$

From Eqs. (176.2) and (176.4),

$$M_1 l = \frac{Qb^3}{l} - Qb^2 \tag{176.5}$$

$$M_1 = -\frac{Qb^2}{l^2}(l - b) = -\frac{Qb^2 a}{l^2} \tag{176.6}$$

By symmetry,

$$M_2 = -\frac{Qa^2b}{l^2} \tag{176.7}$$

Problems

176-1. Calculate the shear at the left end of the beam of Fig. 247.

Ans. $V_1 = Qb^2(l + 2a)/l^3$ (176.8)

176-2. Find the maximum positive moment of the beam of Prob. 176-1. Find the moment when the load is at the middle.

Ans. $M = 2Qa^2b^2/l^3$; $M = Ql/8$.

176-3. What is the moment at each end when the load is at the middle?

Ans. $M_1 = M_2 = -Ql/8$.

176-4. Find the deflection under the load for the beam of Fig. 247. Use the reversed M_p diagram for a simply supported span, as shown in Fig. 245. Calculate the deflection when $a = b$.

Ans. $y = -Qa^3b^3/3EIl^3$; $y = -Ql^3/192EI$.

177. Three Moments. Figure 248 shows a beam which is continuous over three fixed points A, B, and C. The points B and C

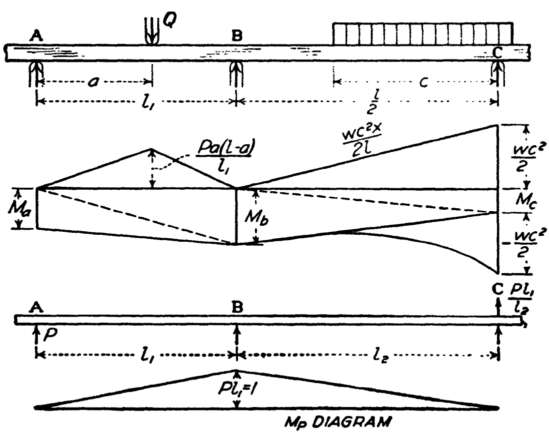


FIG. 248. Three moments.

carry the beam as a simply supported span. The dummy force P is applied to the overhanging end of length l_1 at the point A. The moment of P about B is balanced by the additional reaction of Pl_1/l_2 at C. The M_p diagram has a maximum altitude of Pl_1 at B. Since the $M_p M_q$ term is equated to zero, Pl_1 may conveniently be equated to unity.

$$\frac{M_a l_1}{2} \times \frac{1}{3} + \frac{M_b l_1}{2} \times \frac{2}{3} + \frac{Pa(l-a)}{2} \times \frac{l+a}{3} + \frac{M_b l_2}{2} \times \frac{2}{3} + \frac{M_c l_2}{2} \times \frac{1}{3} - \frac{wc^2 l_2}{4} \times \frac{1}{3} - \frac{wc^3}{6} \times \frac{1}{4} = 0 \tag{177.1}$$

$$M_a l_1 + 2M_b l_1 + 2M_b l_2 + M_c l_2 + Pab^2 + \frac{wc^2 l_2}{2} - \frac{wc^3}{4} = 0 \quad (177.2)$$

178. Closed Ring. Figure 249 shows a closed ring, the radius of which is relatively large compared with the radius of the cross section. This ring is supported at the top and subjected to a load Q at the bottom so that the resultant applied force acts along a diameter. For

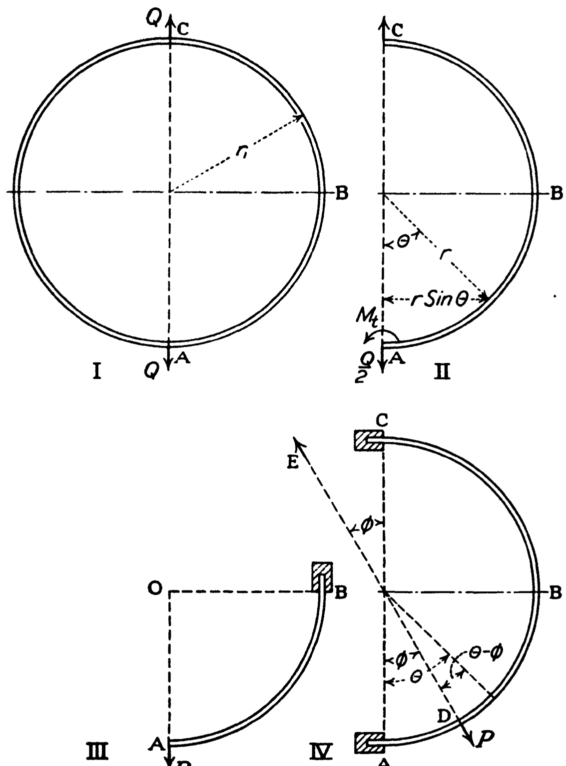


FIG. 249. Closed ring with single load.

beams that are curved in any way, the method of *elastic energy* has great advantages.

In the treatment of deflection of curved beams, forces *outward* along a radius cause positive moment. The moment is *positive* when the *outer* fibers are in compression. Considering the right half of the ring as the free body, there is no horizontal tension at A or C. No horizontal external forces act on this free body; hence the sum of the tensions at A and C is zero. From symmetry, the tension at A must have

the same magnitude and sign as the tension at C. Since the sum is zero, each force must be zero. The vertical load on each half is $Q/2$.

$$M = M_a + \frac{Qr}{2}\sin\theta. \tag{178.1}$$

From symmetry it is evident that there is no change of slope at A, B, or C. A portion from A to B may be regarded as a free body fixed at B. A dummy moment M_t applied at A is transmitted as a constant M_t from A to B (or from A to C). The element of length is $dl = r\,d\theta$.

$$EI\theta = \frac{\int M_t[M_a + (Qr/2)\sin\theta]r\,d\theta}{M_t} \tag{178.2}$$

Since M_t is a constant, it may be canceled.

$$0 = \int_0^{\pi/2}\left(M_a + \frac{Qr}{2}\sin\theta\right)r\,d\theta = r\left[M_a\theta - \frac{Qr}{2}\cos\theta\right]_0^{\pi/2} \tag{178.3}$$

$$\frac{M_a\pi}{2} + \frac{Qr}{2} = 0 \qquad M_a = -\frac{Qr}{\pi} = -0.3183Qr \tag{178.4}$$

From Eq. (178.1),

$$M_b = -\frac{Qr}{\pi} + \frac{Qr}{2} = (-0.3183 + 0.50)Qr = 0.1817Qr \tag{178.5}$$

To find the elongation of the vertical radius OA, the quadrant AB of Fig. 249,II is taken as a free body, which is fixed at B and subjected to the dummy load P at A. $M_p = Pr\sin\theta$.

$$\frac{M_pM_q}{P} = Qr^2\sin\theta\left(-\frac{1}{\pi} + \frac{\sin\theta}{2}\right) = Qr^2\left(-\frac{\sin\theta}{\pi} + \frac{\sin^2\theta}{2}\right) \tag{178.6}$$

$$EIy = Qr^3\int_0^{\pi/2}\left(-\frac{\sin\theta}{\pi} + \frac{1}{4} - \frac{\cos 2\theta}{4}\right)d\theta \tag{178.7}$$

$$EIy = Qr^3\left[\frac{\cos\theta}{\pi} + \frac{\theta}{4} - \frac{\sin 2\theta}{8}\right]_0^{\pi/2} = Qr^3\left(-\frac{1}{\pi} + \frac{\pi}{8}\right) \tag{178.8}$$

$$y = \frac{Qr^3}{EI}(-0.3183 + 0.3927) = 0.0744\frac{Qr^3}{EI} \tag{178.9}$$

which is the elongation of the radius OA. The diameter CA is lengthened $0.1488Qr^3/EI$.

The same result is obtained if the half circle AC is treated as a free body. The M_p expression is not changed but the limits become 0 and π. The integral gives the elongation of the diameter.

To find the elongation of any diameter, such as DE, which makes an

angle ϕ with the vertical, dummy forces of magnitude $2P$ are applied at the ends of this diameter. Each dummy force is supposed to be broken into two equal forces of magnitude P. The portion DC of the ring is supposed to be fixed at C.

$$M_p = Pr \sin(\theta - \phi) = Pr(\sin\theta\cos\phi - \cos\theta\sin\phi).$$

$$M_q = Qr\left(-\frac{1}{\pi} + \frac{\sin\theta}{2}\right)$$

$$\frac{M_pM_q}{P} = Qr^2\left[-\frac{\sin(\theta - \phi)}{\pi} + \frac{\sin^2\theta\cos\phi}{2} - \frac{\sin\theta\cos\theta\sin\phi}{2}\right] \tag{178.10}$$

$$\frac{M_pM_q}{P} = Qr^2\left[-\frac{\sin(\theta - \phi)}{\pi} + \frac{\cos\phi}{4} - \frac{\cos 2\theta\cos\phi}{4} - \frac{\sin 2\theta\sin\phi}{4}\right] \tag{178.11}$$

$$\frac{\int_\phi^\pi M_pM_qr\,d\theta}{P} = Qr^3\left[\frac{\cos(\theta - \phi)}{\pi} + \frac{\theta\cos\phi}{4} - \frac{\sin 2\theta\cos\phi}{8} + \frac{\cos 2\theta\sin\phi}{8}\right]_\phi^\pi \tag{178.12}$$

$$\frac{\int_\phi^\pi M_pM_qr\,d\theta}{P} = Qr^3\left[-\frac{\cos\phi}{\pi} - \frac{1}{\pi} + \frac{(\pi - \phi)\cos\phi}{4} + \frac{\sin 2\phi\cos\phi}{8} + \frac{\sin\phi}{8} - \frac{\cos 2\phi\sin\phi}{8}\right] \tag{178.13}$$

$$\frac{\sin 2\phi\cos\phi}{8} - \frac{\cos 2\phi\sin\phi}{8} = \frac{\sin\phi}{8}.$$

$$\frac{\int_\phi^\pi M_pM_qr\,d\theta}{P} = Qr^3\left[-\frac{1}{\pi} - \frac{\cos\phi}{\pi} + \frac{(\pi - \phi)\cos\phi}{4} + \frac{\sin\phi}{4}\right] \tag{178.14}$$

The portion of the right half of the ring from A to D is supposed to be fixed at A. The dummy moment is the moment of the second half of the force $2P$ acting at D.

$$M_p = P\sin(\phi - \theta).$$

$$\frac{\int_0^\phi M_pM_qr\,d\theta}{P} = Qr^3\int_0^\phi\left[-\frac{\sin(\phi - \theta)}{\pi} + \frac{\sin 2\theta\sin\phi}{4} - \frac{\cos\phi}{4} + \frac{\cos 2\theta\cos\phi}{4}\right]d\theta \tag{178.15}$$

$$\frac{\int_0^\phi M_p M_q r\, d\theta}{P} = Qr^3\left[-\frac{\cos(\phi-\theta)}{\pi} - \frac{\cos 2\theta \sin\phi}{8} - \frac{\theta\cos\phi}{4} + \frac{\sin 2\theta\cos\phi}{8}\right]_0^\phi \quad (178.16)$$

$$= Qr^3\left(-\frac{1}{\pi} + \frac{\cos\phi}{\pi} - \frac{\cos 2\phi\sin\phi}{8} + \frac{\sin\phi}{8} - \frac{\phi\cos\phi}{4} + \frac{\sin 2\phi\cos\phi}{8}\right) \quad (178.17)$$

$$= Qr^3\left(-\frac{1}{\pi} + \frac{\cos\phi}{\pi} + \frac{\sin\phi}{4} - \frac{\phi\cos\phi}{4}\right) \quad (178.18)$$

When Eqs. (178.14) and (178.18) are added to get the total energy in the left half of the ring, the result is

$$EIy = Qr^3\left[-\frac{2}{\pi} + \frac{(\pi - 2\phi)\cos\phi}{4} + \frac{\sin\phi}{2}\right] \quad (178.19)$$

Problems

178-1. Find the elongation of the diameter in the line of the load by means of Eq. (178.19) *Ans.* Elongation $= 0.1488Qr^3/EI$.

178-2. Find the elongation of a diameter at 90° with the line of the load. *Ans.* Elongation $= -0.1366Qr^3/EI$.

178-3. Find the elongation at 45° with the line of the load. *Ans.* Elongation $= -0.0054Qr^3/EI$.

178-4. A ring of circular section, 2 in. in diameter, has a radius of 10 in. Find the maximum positive moment and the maximum negative moment when this ring is subjected to a pull of 1,200 lb. *Ans.* $M = -3{,}820$ in.-lb.; 2,180 in.-lb.

179. Closed Ring with Two Symmetrical Loads. Figure 250 shows a closed ring, suspended at the top, which resists a force Q, normal to the circumference, at an angle β to the right of the vertical downward and an equal force at the same angle to the left of the vertical downward. Since the vertical diameter AC divides the ring and the loads symmetrically, the half ring of Fig. 250,II may be taken as the free body. This half ring may be regarded as fixed at the top and fixed, in so far as rotation is concerned, at the bottom. There is an unknown bending moment M_a and an unknown tension H at A.

$$\overbrace{M_q = M_a - Hr(1-\cos\theta)}^{0\,\text{-----}\,\pi} + \overbrace{Qr\sin(\theta-\beta)}^{\beta\,\text{-----}\,\pi} \quad (179.1)$$

Since the integrals required for finding the unknown quantities are equated to zero, any power of the constant radius which multiplies all terms may be dropped.

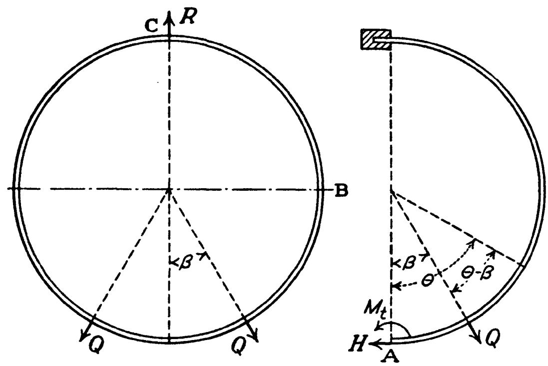


FIG. 250. Closed ring with two loads symmetrically placed.

A dummy moment M_t at A is carried unchanged to the fixed end at C.

$$\frac{\int M_t M_q \, d\theta}{M_t} = \int_0^\pi [M_a - Hr(1 - \cos\theta)] \, d\theta + Qr \int_\beta^\pi \sin(\theta - \beta) \, d\theta \qquad (179.2)$$

$$0 = \Big[M_a\theta - Hr\theta + Hr \sin\theta \Big]_0^\pi - Qr \Big[\cos(\theta - \beta) \Big]_\beta^\pi \qquad (179.3)$$

$$0 = M_a\pi - Hr\pi + Qr\cos\beta + Qr \qquad (179.4)$$

The horizontal force H at A causes no horizontal displacement. A dummy force P at A gives

$$M_p = Pr(1 - \cos\theta).$$

$$\frac{M_p M_q}{Pr} = M_a - M_a\cos\theta - Hr(1 - 2\cos\theta + \cos^2\theta) + Qr\sin(\theta - \beta) - Qr\sin\theta\cos\theta\cos\beta + Qr\cos^2\theta\sin\beta \qquad (179.5)$$

$$\frac{\int M_p M_q \, d\theta}{Pr} = 0 = \int_0^\pi \left[M_a - M_a\cos\theta + Hr\left(-\frac{3}{2} + 2\cos\theta - \frac{\cos 2\theta}{2} \right) d\theta \right] + Qr \int_\beta^\pi \left[\sin(\theta - \beta) - \frac{\sin 2\theta \cos\beta}{2} + \frac{\sin\beta}{2} + \frac{\cos 2\theta \sin\beta}{2} \right] d\theta \qquad (179.6)$$

$$0 = \left[M_a\theta - M_a \sin \theta + Hr\left(-\frac{3\theta}{2} + 2 \sin \theta + \frac{\sin 2\theta}{4}\right)\right]_0^\pi + Qr\left[-\cos(\theta - \beta) + \frac{\cos 2\theta \cos \beta}{4} + \frac{\theta \sin \beta}{2} + \frac{\sin 2\theta \sin \beta}{4}\right]_\beta^\pi \quad (179.7)$$

$$0 = M_a\pi - \frac{3Hr\pi}{2} + Qr\left[\cos \beta + 1 + \frac{\cos \beta}{4} - \frac{\cos 2\beta \cos \beta}{4} + \frac{(\pi - \beta)}{2} \sin \beta - \frac{\sin 2\beta \sin \beta}{4}\right] \quad (179.8)$$

When Eq. (179.4) is subtracted from Eq. (179.8),

$$0 = -\frac{Hr\pi}{2} + Qr\left[\frac{(\pi - \beta) \sin \beta}{2} + \left(\frac{\cos \beta}{4} - \frac{\cos 2\beta \cos \beta}{4} - \frac{\sin 2\beta \sin \beta}{4} = 0\right)\right] \quad (179.9)$$

$$H = \frac{\pi - \beta}{\pi} Q \sin \beta \quad (179.10)$$

Equation (179.10) expresses the remarkably simple relation that the horizontal component of Q is divided in the inverse ratio to the arcs of the circle.

From Eqs. (179.10) and (179.4)

$$M_a = \frac{\pi - \beta}{\pi} Qr \sin \beta - \frac{Qr}{\pi}(1 + \cos \beta) \quad (179.11)$$

Problems

179-1. Find the horizontal tension at the bottom when the angle is 60°. Check by resolutions parallel to Q, using an arc of 120° as the free body.
Ans. $H = Q/\sqrt{3}$.

179-2. In Prob. 179-1, find the horizontal tension at the top.
Ans. $T = Q/2\sqrt{3}$.

179-3. Find H and T when the forces Q make angles of 45° with the vertical.
Ans. $0.5303Q$; $0.1768Q$.

179-4. Solve Prob. 179-2 by horizontal resolutions, with the right half of Fig. 250 as the free body and H known from Prob. 179-1.

179-5. Find M_a when $\beta = 60°$.
Ans. $M_a = (Qr/\sqrt{3}) - (3Qr/2\pi) = 0.0996Qr$.

179-6. With M_a and H_a known, find M_e at the top when $\beta = 60°$. Check by finding M_f at the load Q.
Ans. $M_e = M_f = -0.1888Qr$.

CHAPTER 18

CURVED BEAMS AND HOOKS

180. Stresses in Curved Beams. Figure 251 represents a portion of a curved beam between two planes AB and CD, which are perpendicular to the plane of the paper and intersect at the center of curvature of the beam. The plane AB at the left end of the portion is regarded as fixed, while the plane CD at the right end is rotated through an angle θ to the position $C'D'$ when the beam is bent. The unit stresses in a beam of this kind do not vary directly as the distance from the neutral surface, because the length of the elementary filaments are not the same. If the neutral axis were midway between C and D, the total elongation DD' would equal the total compression CC' but the *unit elongation* at the top would be smaller than the *unit compression* at the bottom because the original length BD is greater than the original length AC.

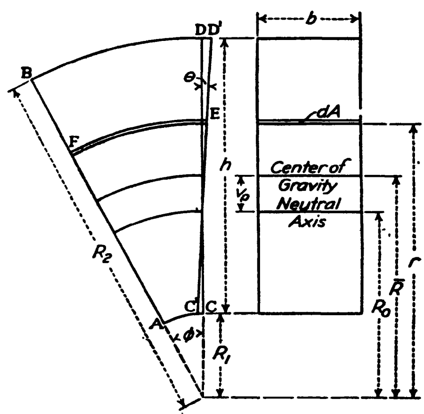


FIG. 251. Curved beam of rectangular section.

Since the length of any filament, such as EF, is proportional to its distance from the center of curvature, the unit deformation and the unit stress vary as the quotient of the distance from the neutral axis divided by the distance from the center of curvature.

In Fig. 251, R_1 is the inside radius, R_2 is the outside radius, R_0 is the radius of the neutral surface, r is the radius of any filament, and v_0 is the distance of the neutral axis from the center of gravity of the cross section. The angle at the center of curvature subtended by the portion of the beam is ϕ, so that the original length of any filament is $r\phi$. The angle through which the plane CD is turned when the beam is bent is θ. If the assumption that a cross section of a beam remains plane when the beam is bent *is valid for curved as well as for straight beams,* the deformation of a filament at a distance $r - R_0$ from the neutral

axis is $(r - R_0)\theta$ and

$$\text{Unit deformation} = \frac{(r - R_0)\theta}{r\phi} \tag{180.1}$$

$$\text{Unit stress} = s = \frac{E(r - R_0)\theta}{r\phi} = k\left(1 - \frac{R_0}{r}\right) \tag{180.2}$$

in which $k = E\theta/\phi$ is a constant for every part of the section at any particular loading.

At the innermost fibers,

$$S_1 = k\left(1 - \frac{R_0}{R_1}\right) \tag{180.3}$$

At the extreme outer fibers,

$$S_2 = k\left(1 - \frac{R_0}{R_2}\right) \tag{180.4}$$

The location of the neutral axis is found by means of the condition that the total force across any section is zero.

$$\text{Force on element of area } dA = k\left(1 - \frac{R_0}{r}\right) dA \tag{180.5}$$

$$\text{Total force} = k \int_{R_1}^{R_2} \left(1 - \frac{R_0}{r}\right) dA = 0 \tag{180.6}$$

$$A = R_0 \int_{R_1}^{R_2} \frac{dA}{r} \qquad R_0 = \frac{A}{\int_{R_1}^{R_2} \frac{dA}{r}} \tag{180.7}$$

The first step in the derivation of a formula for the stress in a curved beam of any section is the location of the neutral axis for that section by means of Eq. (180.7), which does not depend upon the constant k. With R_0 now known, the second step is the derivation of an expression for the resisting moment to be used for the elimination of k.

Since the resisting moment of a beam is a couple, the calculation of the moment of this couple may be made with respect to any axis whatever. The mathematics is greatly simplified if the *origin of moments* is taken at the center of curvature of the beam. With the *origin of coordinates* at this center of curvature, the moment arm of every element is r.

$$\text{Moment} = \int_{R_1}^{R_2} sr\, dA = k \int_{R_1}^{R_2} r\left(1 - \frac{R_0}{r}\right) dA$$

$$= k \int_{R_1}^{R_2} (r - R_0)\, dA \tag{180.8}$$

$$M = k\left(\int_{R_1}^{R_2} r\, dA - R_0 \int_{R_1}^{R_2} dA\right) = k(\bar{R} - R_0)A \tag{180.9}$$

in which R is the radius to the center of gravity of the cross section.

$$M = kv_0A \qquad \text{Formula XXVI}$$

in which $v_0 = R - R_0$ equals the distance of the center of gravity of the cross section from the neutral surface.

181. Curved Beams of Rectangular Section. For a rectangular beam of breadth b and depth h, $dA = b\,dr$ and $A = bh$.

$$R_0 = \frac{bh}{b\int_{R_1}^{R_2}\frac{dr}{r}} = \frac{h}{\log_e \frac{R_2}{R_1}} \tag{181.1}$$

$$v_0 = R_1 + \frac{h}{2} - R_0 \tag{181.2}$$

Example 1

A curved beam of rectangular section is 3 in. wide and 4 in. high. The inner radius is 4 in. Locate the neutral axis and find the stress at the inner and outer surface.

$$R_2 = 8 \text{ in.} \qquad \frac{R_2}{R_1} = 2$$

$$R_0 = \frac{4}{\log_e 2} = \frac{4}{0.69315} = 5.7707 \text{ in.}$$

$$v_0 = 6 - 5.7707 = 0.2293 \text{ in.}$$

$$\frac{v_0}{h} = 0.05732$$

In a rectangular beam of depth equal to the radius of the inner surface, the neutral axis is shifted toward the center of curvature 0.05732 of the depth, or nearly 6 per cent.

To find the stress at any fiber of a rectangular beam in terms of the bending moment, the constant k is eliminated from Formula XXVI and Eq. (180.2).

$$s = k\left(1 - \frac{R_0}{r}\right) \qquad M = kv_0bh$$

$$s = \frac{M\left[1 - \left(\frac{R_0}{r}\right)\right]}{v_0bh} \tag{181.3}$$

To find the unit stress in the outer fibers,

$$S_2 = \frac{M[1 - (5.7707/8)]}{3 \times 4 \times 0.2293} = 0.10127M$$

$$S_1 = \frac{M[(5.7707/4) - 1]}{2.7516} = 0.16089M$$

If S is the unit stress in a straight beam of the same section,

$$S = \frac{M}{8} \qquad \frac{S_2}{S} = 0.8102 \qquad \frac{S_1}{S} = 1.2871$$

It is convenient to express R_1 and R_2 in terms of h as a unit and to take b as unity. For any particular problem, the values of R_0 and v_0 may be multiplied by the height in inches, and the values of S_1 or S_2 may be divided by bh^2. Since v_0 is the difference between relatively large numbers, it is necessary to use five-place tables to ensure three-place accuracy. If good tables of natural logarithms are not available, common logarithms may be used.

$$R_0 = \frac{1}{\log_e \dfrac{R_2}{R_1}} = \frac{0.43429448}{\log_{10} \dfrac{R_2}{R_1}}$$

Example 2

If $h/R_1 = 2$, find the stress in the outer fibers in terms of the moment regarding h and b as unity.

$$R_1 = \frac{1}{2} \qquad R_2 = \frac{3}{2} \qquad R = 1 \qquad \frac{R_2}{R_1} = 3$$

$$R_0 = \frac{0.43429448}{0.47712125} = 0.91023926h[1]$$

$$v_0 = 0.08976074h$$

$$S_1 = \frac{0.820478}{v_0} = \frac{9.1406M}{bh^2} \qquad \frac{S_1}{S} = 1.5234$$

$$S_2 = \frac{0.393174}{v_0} = \frac{4.3803M}{bh^2} \qquad \frac{S_2}{S} = 0.7300$$

TABLE 19. DISPLACEMENT OF NEUTRAL SURFACE AND RELATIVE STRESSES IN EXTREME FIBERS OF CURVED BEAM OF RECTANGULAR SECTION

Ratio of depth to inner radius, h/R_1	Distance of neutral axis		Ratio of unit stress in extreme fibers to stress in straight beam	
	From center of curvature, R_0/h	From center of gravity, v_0/h	S_1/S	S_2/S
0.50	2.466303	0.033697	1.1532	0.8799
1.00	1.442695	0.057305	1.2875	0.8103
1.50	1.091357	0.075310	1.4098	0.7639
2.00	0.910239	0.089761	1.5234	0.7300
3.00	0.721347	0.111986	1.7324	0.6814
4.00	0.621335	0.128665	1.9240	0.6514
5.00	0.558111	0.141889	2.1020	0.6282

[1] The logarithm of 3 in the expression for R_0 was taken from "Logarithmic and Trigonometric Tables," rev. ed., p. 133, edited by Earl Raymond Hedrick, The Macmillan Company, New York, 1931.

Table 19 gives the ratio of the stress in the extreme fibers of a curved beam to the corresponding stress in a straight beam. These ratios are plotted as ordinates on Fig. 252. The curve for the unit stress at the concave surface does not differ greatly from the broken straight line, for which the equation is

$$S_1 = S\left(1 + 0.25\,\frac{h}{R_1}\right) \qquad S_1 = \frac{6M}{bh^2}\left(1 + 0.25\,\frac{h}{R_1}\right) \tag{181.4}$$

Problems

181-1. Verify Table 19 for $h/R_1 = 1.5$.

181-2. A curved cast-steel beam is 5 in. wide, has an inner radius of 2 in. and an outer radius of 8 in. By means of the table, find the unit stress at the inner and outer fibers under a bending moment of 20,000 ft-lb. Find the unit stress at 1 in. from the outer surface by means of Eq. (181.3).

Ans. 13,859 psi; 5,451 psi; 4,545 psi.

181-3. A cast-steel beam is 6 in. wide, has an inner radius of 2 in. and an outer radius of 7 in. Find the stress in the extreme fibers when $M = 9{,}000$ ft-lb. Solve approximately by Eq. (181.4). Compare with the true curve of Fig. 252.

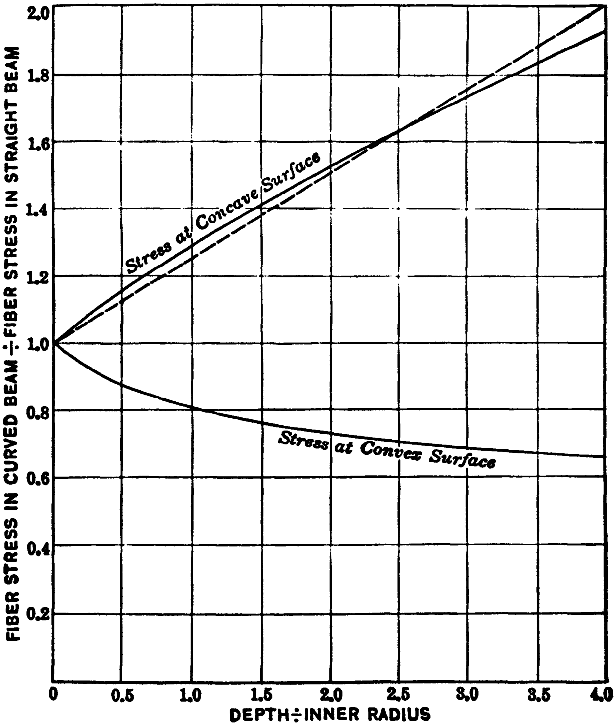


FIG. 252. Stress at outer fibers of curved beam of rectangular section.

Example 3

Find the stress in the extreme fibers of the section of Fig. 253. Compare with the stress in a straight beam.

$$A = 20 \text{ in.}^2 \qquad \bar{R} = 6.2 \text{ in.}$$

$$\int \frac{dA}{r} = 6\left[\log_e r\right]_4^6 + 2\left[\log_e r\right]_6^{10} = 3.45444 \qquad R_0 = \frac{20}{3.45444} = 5.78965 \text{ in}$$

$$v_0 = 0.41035 \qquad S_1 = \frac{1.45582 - 1}{8.2070} M = 0.054515M \qquad S_2 = ?$$

Problems

181-4. A hollow beam is 1 in. square outside and 0.8 in. square inside. The beam is curved to an inner radius of 0.5 in. Find the ratio of the stress in the outer fibers to the stress in a straight beam.

Ans. $S_1/S = 1.3458$; $S_2/S = 0.8131$.

181-5. Solve Prob. 181-4 if the inside is 0.6 in. square.

Ans. $S_1/S = 1.4372$; $S_2/S = 0.7811$.

181-6. Solve Prob. 181-4 if the inside is 0.4 in. square.

Ans. $S_1/S = 1.4918$; $S_2/S = 0.9547$.

181-7. Compare the answers of the last three problems with a solid rectangular section for which h/R_1 is the same.

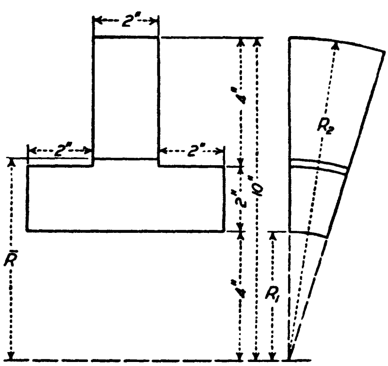


FIG. 253. Curved beam of T section.

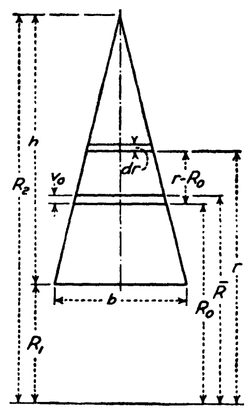


FIG. 254. Curved beam of triangular section.

182. Triangular Beam. Figure 254 shows a triangular beam with its base at a distance R_1 from the center of curvature.

$$dA = \frac{b}{h}(R_2 - r)\,dr \qquad s = k\left(1 - \frac{R_0}{r}\right)$$

$$\int_{R_1}^{R_2} s\,dA = \frac{bk}{h}\int_{R_1}^{R_2}\left(1 - \frac{R_0}{r}\right)(R_2 - r)\,dr$$

$$= \frac{bk}{h}\int_{R_1}^{R_2}\left(R_2 - r + R_0 - \frac{R_0R_2}{r}\right)dr \qquad (182.1)$$

$$0 = \left[R_2 r - \frac{r^2}{2} + R_0 r - R_0 R_2 \log_e r \right]_{R_1}^{R_2}$$

$$= R_2 h - \frac{R_2^2 - R_1^2}{2} + R_0 h - R_0 R_2 \log_e \frac{R_2}{R_1} \tag{182.2}$$

$$R_0 = -\frac{h_2}{2\left(h - R_2 \log \frac{R_2}{R_1}\right)} = \frac{h}{2\left(\frac{R_2}{h} \log_e \frac{R_2}{R_1} - 1\right)} \tag{182.3}$$

Example

Locate the neutral surface in a curved beam of triangular section for which $h/R_1 = 5$. Find the ratio of the stress in the outer fibers to the stress in a straight beam of the same section.

$$R_1 = 0.2h \qquad R_2 = 1.2h \qquad \frac{R_2}{R_1} = 6$$

$$1.2 \times 1.79175947 - 1 = 1.150111364$$

$$R_0 = \frac{0.5h}{1.150111364} = 0.4347405h$$

$$\bar{R} = \underline{0.5333333h}$$

$$v_0 = \overline{0.0985928h}$$

$$S_2 = \frac{[1 - (0.4347405/1.2)]M}{(bh/2) \times 0.0985928h} = \frac{12.9364M}{bh^2}$$

$$\frac{S_2}{S} = \frac{12.9364M}{bh^2} \times \frac{bh^2}{24M} = 0.5390$$

$$S_1 = \frac{[(0.4347405/0.2) - 1]M}{(bh/2) \times 0.0985928h} = \frac{23.8089M}{bh^2}$$

$$\frac{S_1}{S} = \frac{23.8089M}{bh^2} \times \frac{bh^2}{12M} = 1.9841$$

TABLE 20. DISPLACEMENT OF NEUTRAL SURFACE AND RELATIVE STRESSES IN EXTREME FIBERS OF CURVED BEAM OF TRIANGULAR SECTION

Ratio of depth to inner radius, h/R_1	Distance of neutral axis from		Stress in extreme fibers in terms of M/bh^2		Ratio of stress to stress in straight beam	
	Center of curvature, R_0/h	Center of gravity, v_0/h	S_1	S_2	S_1/S	S_2/S
0.5	2.310586	0.022747	13.6540	20.2053	1.1378	0.8419
1.0	1.294350	0.038983	15.1014	18.1015	1.2584	0.7542
1.5	0.948495	0.051505	16.4156	16.7325	1.3680	0.6972
2.0	0.771702	0.061631	17.6341	15.7561	1.4695	0.6565
3.0	0.589350	0.077317	19.8675	14.4337	1.6556	0.6014
4.0	0.494170	0.089163	21.9079	13.5631	1.8257	0.5651
5.0	0.434740	0.098593	23.8089	12.9364	1.9841	0.5390

Problems

182-1. Make the calculations of Table 20 for $h/R_1 = 2$, using five-place natural logarithms to determine R_0.

182-2. Make the calculations of Table 20 for $h/R_1 = 4$, using five-place common logarithms.

182-3. A curved beam of triangular section is 4 in. high and 3 in. wide. The inner radius from the center of curvature to the base of the triangle is 12 in. Find the unit stress in the extreme fibers for a bending moment of 1,500 ft-lb.

182-4. Plot curves for the ratios of Tables 19 and 20 on the same sheet. What can you learn from these curves?

183. Curved Beams of Converging Trapezoidal Section. Figure 255 shows a curved beam of trapezoidal section with the outer breadth

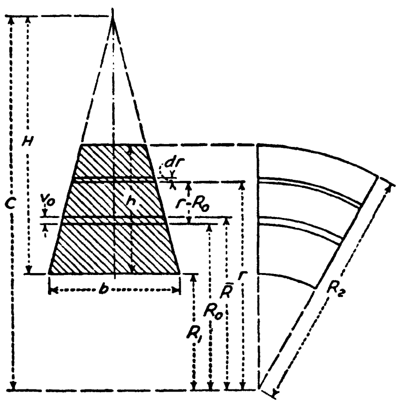


FIG. 255. Curved beam of trapezoidal section, sides converging.

smaller than the inner breadth. The height from the inner surface to the outer surface is h. The height from the inner surface to the intersection of the sides is H; and the radius from the center of curvature to this intersection is C.

$$\text{Total force} = k\int\left(1 - \frac{R_0}{r}\right)dA = k\left(\int_{R_1}^{R_2} dA - R_0\int_{R_1}^{R_2}\frac{dA}{r}\right)$$

$$dA = \frac{b}{H}(C - r)\,dr$$

$$0 = \int_{R_1}^{R_2} dA - \frac{R_0 b}{H}\int_{R_1}^{R_2}\left(\frac{C}{r} - 1\right)dr = A - \frac{R_0 b}{H}\left[C\log_e\frac{R_2}{R_1} - h\right]_{R_1}^{R_2}$$

$$R_0 = \frac{AH}{bh\left(\dfrac{C}{h}\log_e\dfrac{R_2}{R_1} - 1\right)}$$

TABLE 21. DISPLACEMENT OF NEUTRAL SURFACE AND RELATIVE STRESSES IN EXTREME FIBERS OF CURVED BEAM OF TRAPEZOIDAL SECTION WITH OUTER BASE ONE-HALF OF INNER BASE

$$\text{Area} = \frac{3bh}{4} \qquad \bar{y} = \frac{4h}{9} \qquad I_c = \frac{13bh^3}{216} \qquad Z_1 = \frac{13bh^2}{96} \qquad Z_2 = \frac{13bh^2}{120}$$

Ratio of depth to inner radius, h/R_1	Distance of neutral axis from		Stress in extreme fibers in terms of M/bh^2		Ratio of stress to stress in straight beam	
	Center of curvature, R_0/h	Center of gravity, v_0/h	S_1	S_2	S_1/S	S_2/S
0.5	2.412117	0.032327	8.4989	8.0825	1.1509	0.8756
1.0	1.389607	0.054837	9.4731	7.4207	1.2828	0.8039
1.5	1.039183	0.071928	10.3580	6.9786	1.4026	0.7560
2.0	0.858845	0.085599	11.1791	6.6580	1.5138	0.7213
3.0	0.671235	0.106543	12.6860	6.2144	1.7179	0.6732
4.0	0.572249	0.122195	14.0649	5.9162	1.9047	0.6409
5.0	0.509879	0.134565	15.3521	5.6984	2.0789	0.6173

Example

A curved beam of trapezoidal section has an inner base of 2 in., an outer base of 1 in., a height of 3 in., and an inner radius of 1 in. Find the location of the neutral surface, the stress in the extreme fibers in terms of the bending moment, and the ratio of these stresses to the stresses in a straight beam of the same section.

$$R_1 = 1 \qquad R_2 = 4 \qquad C = 7 \qquad H = 6$$

Using common logarithms,

$$R_0 = \frac{4.5 \times 6 \times 0.43429448}{2 \times 3(7/3 \times 0.6020600 - 0.43429448)} = \frac{1.95432516}{0.97051219}$$

$$\bar{R} = 2.333333 \qquad I_c = 1\tfrac{3}{4} \qquad Z_1 = \frac{1\frac{3}{4}}{\frac{4}{3}} = \frac{39}{16}$$

$$R_0 = 2.013705$$

$$v_0 = 0.319628 \qquad Z_2 = \frac{1\frac{3}{4}}{\frac{5}{3}} = \frac{39}{20}$$

$$1 - \frac{2.013705}{4} = 0.496574$$

$$S_2 = \frac{0.496574M}{4.5 \times 0.319628} = 0.345244M$$

$$\frac{S_2}{S} = \frac{0.345244M}{20M/39} = 0.6732$$

$$\frac{2.013705}{1} - 1 = 1.013705 \qquad S_1 = \frac{1.013705M}{1.438326} = 0.704781M$$

$$\frac{S_1}{S} = \frac{0.704781M}{16M/39} = 1.7179$$

Problems

183-1. A curved beam of trapezoidal section has an inner radius of 3 in., inner base of 2 in., outer radius of 6 in., and outer base of 1 in. Find the unit stress at the inner and outer surfaces for a moment of 2,000 ft-lb by means of Table 21 without using the stress in a straight beam.

Ans. $S_1 = 12{,}630$ psi; $S_2 = 9{,}890$ psi.

183-2. The inner radius of a trapezoidal curved beam is equal to the height, and the inner base is four times the outer base. Find the expression for the stresses in the extreme fibers and the ratios of these stresses to the stresses in a straight beam.

Ans. $S_1 = 11.1668M/bh^2$; $S_2 = 10.37351M/bh^2$; $S_1/S = 1.2795$; $S_2/S = 0.7925$.

184. Curved Beam of Diverging Trapezoidal Section. Figure 256 shows a trapezoidal section which is wider at the convex than at the concave surface. The distance C from the center of curvature to the vertex of the triangle is less than R_1.

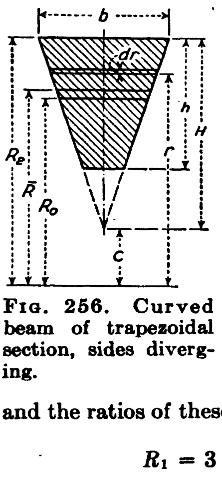


FIG. 256. Curved beam of trapezoidal section, sides diverging.

$$dA = \frac{b}{H}(r - C)\,dr$$

$$R_0 = \frac{AH}{bh\left[1 - \frac{C}{h}\log_e \frac{R_2}{R_1}\right]} \tag{184.1}$$

Example 1

A curved beam of trapezoidal section has an inner radius of 3 in., outer radius of 6 in., inner base of 1 in., and outer base of 4 in. Solve for the stresses in the extreme fibers and the ratios of these to the stresses in a straight beam.

$$R_1 = 3 \qquad R_2 = 6 \qquad \bar{R} = 4.8 \qquad A = 7.5 \qquad b = 4$$

$$h = 3 \qquad H = 4 \qquad C = 2$$

$$R_0 = \frac{7.5 \times 4}{4 \times 3(1 - \frac{2}{3}\log_e 2)} = \frac{2.5}{0.53790188} = 4.647679$$

$$v_0 = 4.8 - 4.647679 = 0.152321 \qquad S_1 = 0.48079M$$

$$\frac{S_1}{S} = 0.48079 \times \frac{11}{4} = 1.3222 \qquad S_2 = 0.19730M$$

$$\frac{S_2}{S} = 0.19730 \times \frac{33}{8} = 0.8139$$

Example 2

Solve Example 1 if the breadth at the inner surface is changed to 2 in. and all other data are unchanged. The inclined surfaces intersect on the center of curvature. Since the width increases directly as r, the neutral surface is midway between the inner and outer surfaces.

$$R_0 = 4.5 \text{ in.} \qquad v_0 = \frac{1}{6} \text{ in.} \qquad S_1 = \frac{M}{3}$$

$$\frac{S_1}{S} = \frac{M}{3} \times \frac{39}{10M} = 1.3 \qquad S_2 = \frac{M}{6} \qquad \frac{S_2}{S} = \frac{M}{6} \times \frac{39}{8M} = 0.8125$$

Problems

184-1. Solve Example 1 if the breadth at the inner surface is 2.4 in. and all the other dimensions remain unchanged.

Ans. $R_1 = 4.45575$ in.; $v_0 = 0.16925$ in.; $S_1/S = 1.2957$.

184-2. A beam of trapezoidal section has an outer radius of 6 in., an inner radius of 2 in., an outer breadth of 5 in., and an inner breadth of 3 in. Find the ratios of the stresses in the extreme fibers to the stresses in a straight beam of the same section.

185. Curved Beams of Circular Section.

Figure 257 shows a circular section of diameter D with its center at a distance c from the center of curvature. The expressions for finding R_0 are

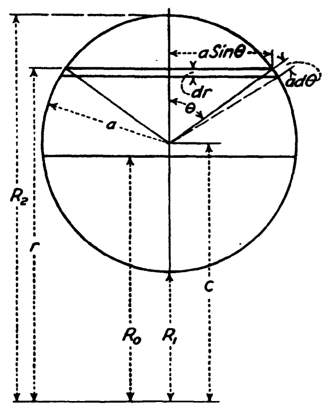


FIG. 257. Curved beam of circular section.

$$dA = -2a \sin \theta \, dr$$
$$r = c + a \cos \theta$$
$$dr = -a \sin \theta \, d\theta$$
$$dA = 2a \sin^2 \theta \, d\theta$$

in which a is the radius of the circle and θ is the angle which the radius to the end of the element makes with the vertical upward.

$$\frac{dA}{r} = \frac{2a^2 \sin^2 \theta \, d\theta}{c + a \cos \theta} \tag{185.1}$$

$$\frac{dA}{r} = \left[-2a \cos \theta + 2c - \frac{2(c^2 - a^2)}{c + a \cos \theta} \right] d\theta \tag{185.2}$$

$$\int \frac{dA}{r} = \left[-2a \sin \theta + 2c\theta + 2\sqrt{c^2 - a^2} \left(\sin^{-1} \frac{a + c \cos \theta}{c + a \cos \theta} \right) \right]_0^{\pi} \tag{185.3}$$

$$= 2c\pi + 2\sqrt{c^2 - a^2} \left(\sin^{-1} \frac{a - c}{c - a} - \sin^{-1} \frac{a + c}{c + a} \right) \tag{185.4}$$

$$= 2c\pi - 2\pi\sqrt{c^2 - a^2} = 2\pi(c - \sqrt{c^2 - a^2}) \tag{185.5}$$

$$R_0 = \frac{\pi a^2}{2\pi(c - \sqrt{c^2 - a^2})} = \frac{c + \sqrt{c^2 - a^2}}{2} \tag{185.6}$$

$$v_0 = c - R_0 = \frac{c - \sqrt{c^2 - a^2}}{2} \tag{185.7}$$

Example 1

A curved beam of circular section is 4 in. in diameter and the radius of curvature at the inner surface is 2 in. Find the displacement of the neutral surface, the unit stress at the inner surface and at the outer surface, and the ratio of these stresses to the stress in the extreme fibers of a straight beam.

$$R_0 = \frac{4 + \sqrt{16 - 4}}{2} = 2 + \sqrt{3} = 3.73205 \qquad v_0 = 2 - \sqrt{3} = 0.26795$$

$$S_1 = \frac{[(R_0/2) - 1]M}{4\pi(2 - \sqrt{3})} = \frac{\sqrt{3}(2 + \sqrt{3})M}{8\pi} = \frac{2(\sqrt{3} + 3)M}{8\pi} = \frac{0.80801M}{\pi}$$

For a straight beam of circular section 4 in. in diameter,

$$S = \frac{M}{2\pi} \qquad \frac{S_1}{S} = 1.6160$$

Problems

185-1. Solve Example 1 for S_2 and S_2/S.

185-2. Solve Example 1 if the inner radius is 1 instead of 2 in. with all the other data given remaining unchanged. Compare with Table 22.

185-3. Solve Example 1 if the inner radius is 4 in. and all the other data remain unchanged. Compare with Table 22.

Since $c = (R_1 + R_2)/2$ and $a = (R_2 - R_1)/2$ (185.8)

$$R_0 = \frac{R_1 + 2\sqrt{R_1R_2} + R_2}{4} = \frac{(\sqrt{R_1} + \sqrt{R_2})^2}{4} \tag{185.9}$$

$$v_0 = \frac{R_1 - 2\sqrt{R_1R_2} + R_2}{4} = \frac{(\sqrt{R_1} - \sqrt{R_2})^2}{4} \tag{185.10}$$

$$S_1 = \frac{(R_0 - R_1)M}{\pi a^2 v_0 R_1} = \frac{(-3R_1 + 2\sqrt{R_1R_2} + R_2)M}{\pi a^2 R_1(R_1 - 2\sqrt{R_1R_2} + R_2)} \tag{185.11}$$

$$S_2 = \frac{(3R_2 - 2\sqrt{R_1R_2} - R_1)M}{\pi a^2 R_2(R_1 - 2\sqrt{R_1R_2} + R_2)} \tag{185.12}$$

Since $S = 4M/\pi a^3$, for a straight beam,

$$\frac{S_1}{S} = \frac{(-3R_1 + 2\sqrt{R_1R_2} + R_2)a}{4R_1(R_1 - 2\sqrt{R_1R_2} + R_2)} \tag{185.13}$$

$$\frac{S_2}{S} = \frac{(3R_2 - 2\sqrt{R_1R_2} - R_1)a}{4R_2(R_1 - 2\sqrt{R_1R_2} + R_2)} \tag{185.14}$$

These equations make it possible to solve for the ratio of the stress in the extreme fibers to the stress in a straight beam by direct substitution.

Example 2

Find the ratio of the unit stress in the extreme fibers to the maximum stress in a straight beam when the inner radius is two-thirds the diameter.

$$R_1 = \frac{2D}{3} \qquad R_2 = \frac{5D}{3} \qquad a = \frac{D}{2}$$

To simplify the calculations, let $R_1 = 2$, $R_2 = 5$, and $a = \frac{3}{2}$.

$$\frac{S_1}{S} = \frac{(-6 + 2\sqrt{10} + 5)\frac{3}{2}}{4 \times 2(2 - 2\sqrt{10} + 5)} = \frac{3(-1 + 2\sqrt{10})}{16(7 - 2\sqrt{10})}$$

$$= \frac{3(-1 + 2\sqrt{10})(7 + 2\sqrt{10})}{16(49 - 40)}$$

$$\frac{S_1}{S} = \frac{-7 + 12\sqrt{10} + 40}{48} = \frac{70.9473}{48} = 1.47807$$

$$\frac{S_2}{S} = \frac{51 + 12\sqrt{10}}{120} = 0.74123$$

Figure 258 is plotted from Table 22. The relative stress at the concave surface does not differ largely from that represented by the straight line

$$S_1 = S\left(1 + 0.3\frac{D}{R_1}\right) = \frac{4M}{\pi a^3}\left(1 + 0.3\frac{D}{R_1}\right) \qquad (185.15)$$

The calculations by Eqs. (185.11) and (185.12) are so easy that it is hardly worth while to use this approximate relation.

Problems

185-4. Find the unit stress in the extreme fibers of a curved beam which is 2.5 in. in diameter and has an inner radius of 2 in. when the moment is 15,708 in.-lb.

Ans. $S_1 = 14{,}400$ psi; $S_2 = 7{,}822$ psi.

185-5. A curved beam, 4 in. in diameter, has an inner radius of 1 in. Find the moment which will give unit stress of 1,200 psi at the inner fibers. What will be the stress at the outer fibers?

TABLE 22. RELATIVE STRESSES IN EXTREME FIBERS OF CURVED BEAMS OF CIRCULAR SECTION

Ratio of diameter to inner radius, D/R_1	Distance of neutral axis from		Ratio of stress to stress in a straight beam	
	Center of curvature, R_0/D	Center of gravity, v_0/D	S_1S	S_2S
0.5	2.474745	0.025355	1.17487	0.86658
1.0	1.457106	0.042893	1.33211	0.79105
1.5	1.110380	0.056287	1.47807	0.74123
2.0	0.933013	0.066987	1.61603	0.70534
3.0	0.750000	0.083333	1.87500	0.65625
4.0	0.654508	0.095492	2.11803	0.62361
5.0	0.594950	0.105050	2.34974	0.59996

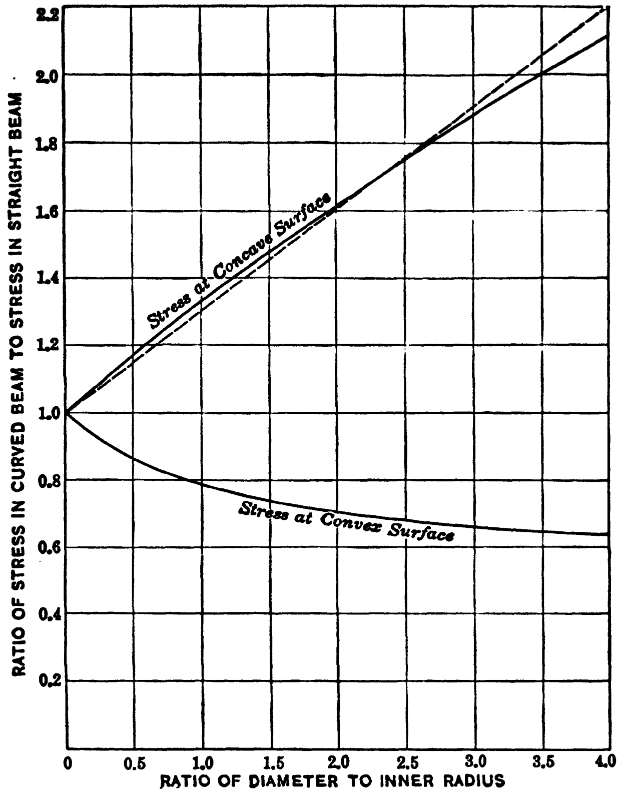


FIG. 258. Unit stress in outer fibers of curved beam of circular section.

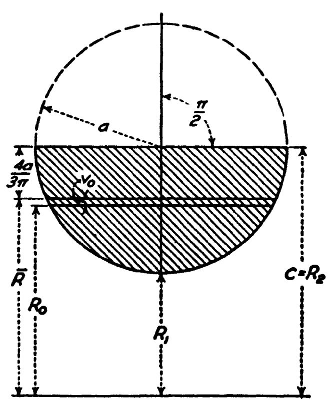


FIG. 259. Semicircular curved beam, diameter on convex surface.

186. Curved Beams of Semicircular Section. Figure 259 shows a semicircular section with the diameter which bounds it on the convex surface of the beam. The limits of θ in Eq. (185.3) are $\pi/2$ and π.

$$\left[\sin^{-1}\frac{a + c\cos\theta}{c + a\cos\theta}\right]_{\pi/2}^{\pi} = \sin^{-1}\frac{a - c}{c - a} - \sin^{-1}\frac{a}{c} \tag{186.1}$$

$$\int_{\pi/2}^{\pi}\frac{dA}{r} = 2a + c\pi - 2\sqrt{c^2 - a^2}\left(\frac{\pi}{2} + \sin^{-1}\frac{a}{c}\right) \tag{186.2}$$

Example 1

Find the displacement of the neutral surface, the stress in the extreme fibers, and the ratio of these stresses to the stress in a straight beam for a curved beam of semicircular section with the radius of curvature of the inner surface equal to radius of the section.

$$a = 1 \qquad c = 2 \qquad A = 0.5\pi a^2 \qquad \sin^{-1} 0.5 = \frac{\pi}{6}$$

$$R_0 = \frac{0.5\pi a^2}{2 + 2\pi - \sqrt{3}(4\pi/3)} = \frac{0.5\pi a}{(0.63662 + 2 - 2.30940)\pi} \tag{186.3}$$

$$R_0 = \frac{0.5a}{0.32722} = 1.52802a$$

$$\bar{R} = \left(2 - \frac{4}{3\pi}\right)a = 1.57559a \qquad v_0 = 0.04757a \qquad S_1 = \frac{7.0664M}{a^3}$$

At the convex surface of the *section* of a straight beam,

$$S = \frac{M}{0.19069a^3} \qquad \frac{S_1}{S} = 7.0664 \times 0.19069 = 1.3475$$

$$S_2 = \frac{3.1582M}{a^3} \qquad \frac{S_2}{S} = 3.1582 \times 0.25861 = 0.8167$$

Compare these ratios with previous tables for the same h/R_1.

Problems

186-1. Solve the preceding example for $a = 2R_1$.
Ans. $v_0 = 0.04725a$; $S_1 = 8.59704M/a^3$; $S_1/S = 1.6923$; $S_2 = 2.8504M/a^3$; $S_2/S = 0.7372$.

186-2. Solve the preceding example for $a = 4R_1$.

Figure 260 shows a semicircular section with the diameter which bounds it on the concave surface of the curved beam. The limits of θ in Eq. (185.3) are 0 and $\pi/2$.

$$\left[\sin^{-1}\frac{a + c\cos\theta}{c + a\cos\theta}\right]_{0}^{\pi/2} = \sin^{-1}\frac{a}{c} - \sin^{-1}\frac{a + c}{c + a}$$

$$\int_{0}^{\pi/2}\frac{dA}{r} = -2a + c\pi - 2\sqrt{c^2 - a^2}\left(\frac{\pi}{2} - \sin^{-1}\frac{a}{c}\right) \tag{186.4}$$

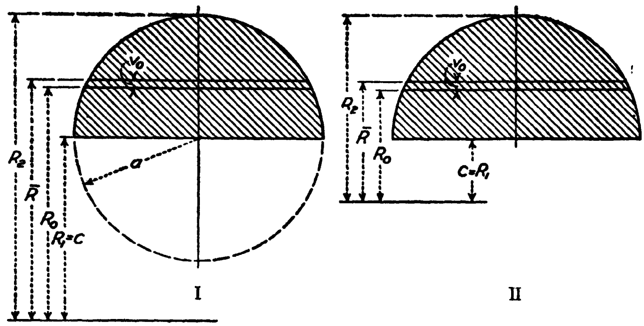


FIG. 260. Semicircular curved beam, diameter on concave surface.

Example 2

A curved beam of semicircular section has the diameter of each section on the concave surface of the beam at a distance $2a$ from the center of curvature. Find the ratio of the stresses in the extreme fibers to the stresses in a straight beam.

$$c = R_1 = 2a \qquad A = 0.5\pi a^2 \qquad \sin^{-1} 0.5 = \frac{\pi}{6}$$

$$R_0 = \frac{0.5\pi a}{-2 + 2\pi - \sqrt{3}\,2\pi/3} = \frac{0.5\pi a}{(-0.63662 + 2 - 1.15470)\pi}$$

$$R_0 = \frac{0.5a}{0.20868} = 2.39601a \qquad v_0 = 2.42441a - R_0 = 0.02840a$$

$$S_1 = \frac{0.198005M}{0.5\pi a^2 \times 0.02840a} = \frac{4.4600M}{a^3} \qquad \frac{S_1}{S} = 1.1536$$

$$S_2 = \frac{0.201663M}{0.0446107a^3} = \frac{4.5205M}{a^3} \qquad \frac{S_2}{S} = 0.8610$$

Problems

186-3. A curved beam similar to Fig. 260 has an inner radius of length a. Solve for the stresses in the extreme fibers.

Ans. $v_0 = 0.04844a$; $S_1/S = 1.2779$; $S_2/S = 0.7832$.

186-4. Solve Prob. 186-3 if the inner radius is $5a/4$.

Ans. $R_0 = 1.63325a$; $v_0 = 0.04116a$; $S_1 = 4.7422M/a^3$; $S_1/S = 1.2264$; $S_2 = 4.5205M/a^3$; $S_2/S = 0.8087$.

When a is greater than c (Fig. 260,II) the expression $\sqrt{c^2 - a^2}$ in Eq. (185.3) becomes imaginary. For a greater than c,

$$\frac{d\theta}{c + a\cos\theta} = \frac{1}{\sqrt{a^2 - c^2}} \log_e \frac{a + c\cos\theta + \sqrt{a^2 - c^2}\sin\theta}{c + a\cos\theta} \qquad (186.5)^{1}$$

$$\int \frac{dA}{r} = -2a\sin\theta + 2c\theta$$

$$+ 2\sqrt{a^2 - c^2}\log_e \frac{a + c\cos\theta + \sqrt{a^2 - c^2}\sin\theta}{c + a\cos\theta} \qquad (186.6)$$

[1] PEIRCE, B. O., "Short Table of Integrals," abridged ed., p. 22, Ginn and Company, Boston, 1914.

For the semicircle of Fig. 260, the limits are 0 and $\pi/2$.

$$\int_0^{\pi/2} \frac{dA}{r} = -2a + c\pi + 2\sqrt{a^2 - c^2}\left(\log_e \frac{a + \sqrt{a^2 - c^2}}{c} - \log_e \frac{a + c}{c + a}\right) \quad (186.7)$$

$$\int_0^{\pi/2} \frac{dA}{r} = -2a + c\pi + 2\sqrt{a^2 - c^2}\log_e \frac{a + \sqrt{a^2 - c^2}}{c} \quad (186.8)$$

Example 3

Find the stresses in the extreme fibers of the beam of Fig. 260 for $c = 0.5a$.

$$\log_e (2 + \sqrt{3}) = \log_e 3.73205 = 1.316944$$

$$\sqrt{3} \times 1.316944 = 2.280812 \qquad 2.280812 + 1.570796 - 2 = 1.851606$$

$$\frac{1.570796a^2}{1.851606a} = R_0 = 0.84835a \qquad v_0 = 0.92441a - 0.84835a = 0.07606a$$

$$\frac{0.84835a}{0.5a} - 1 = 0.69676 \qquad S_1 = \frac{0.69676M}{0.119477a^3} = \frac{5.8318M}{a^3}$$

$$\frac{S_1}{S} = 5.8318 \times 0.25861 = 1.50816$$

187. Hooks. A hook is equivalent to a curved beam which is subjected to eccentric tension. As in a short block which is eccentrically loaded, the total pull P may be replaced by a pull P through the center of gravity of the section and a bending moment Pe, in which e is the distance of the center of gravity of the section from the line of the load. The direct tensile stress is P/A. If the hook is straight at the section under consideration, the tensile stress which is due to bending is Pev/I and the stress at the innermost fibers is

$$S_t = \frac{P}{A}\left(1 + \frac{ec}{r^2}\right) \quad (187.1)$$

At the outermost fibers, if the hook is straight at the section under consideration,

$$S = \frac{P}{A}\left(1 - \frac{ec}{r^2}\right) \quad (187.2)$$

Since the eccentricity of the load on a hook is always so great that ec/r^2 is greater than unity, S is a compressive stress.

While Eqs. (187.1) and (187.2) afford an *approximate* method of finding the maximum stresses in a hook or curved bar which is subjected to tension or compression, there is an error on the side of danger, unless the curvature at the section under consideration is relatively small.

For accurate results, it is necessary to regard the hook as a curved beam in calculating the bending stress. At the innermost fibers of a hook,

$$S_t = \frac{P}{A} + S_1 \qquad (187.3)$$

in which S_1 is the unit stress at the concave surface calculated for a curved beam with the moment Pe. At the outermost fibers,

$$S_c = S_2 - \frac{P}{A} \qquad (187.4)$$

in which S_2 is the bending stress at the convex surface calculated for a curved beam with moment Pe.

188. Curved Bars of Rectangular Section. While hooks are not made with rectangular sections, curved bars frequently have this form.

Example

A curved bar of rectangular section is 2 in. wide. At the section farthest from the applied load the inner radius is 3 in. and the outer radius is 6 in. The load is 3,000 lb tension and passes through the center of curvature. Find the maximum unit tensile and compressive stress at the most remote section.

$$R_1 = 3 \text{ in.} \qquad R_2 = 6 \text{ in.} \qquad h = 3 \text{ in.} \qquad \frac{h}{R_1} = 1$$

$$\frac{P}{A} = \frac{3{,}000}{6} = 500 \text{ psi tension} \qquad M = 3{,}000 \times 4.5 = 13{,}500 \text{ in.-lb moment}$$

For a straight beam,

$$S = \frac{13{,}500}{3} = 4{,}500 \text{ psi}$$

Since $h/R_1 = 1$, the ratios S_1/S and S_2/S may be taken from Table 19 without interpolating.

$$S_1 = 4{,}500 \times 1.2876 = 5{,}794 \text{ psi tension}$$

$$\text{Total tension at concave surface} = 5{,}794 + 500 = 6{,}294 \text{ psi}$$

$$S_2 = 4{,}500 \times 0.8104 = 3{,}647 \text{ psi compression.}$$

$$\text{Total compression at convex surface} = 3{,}647 - 500 = 3{,}147 \text{ psi.}$$

Problems

188-1. A curved beam 2 in. square has an inner radius of 3 in. and an outer radius of 5 in. It is subjected to a tension of 2,000 lb which passes through the center of curvature. Find the tensile stress at the inner surface. Calculate S_1 from the equations of Art. 181, and make an approximate solution by interpolating Table 19. *Ans.* $S_t = 7{,}698$ psi.

188-2. A curved bar of rectangular section is 3 in. wide. The inner radius is 6 in. and the outer radius is 10 in. The load is 6,000 lb and the line of the load is 3 in. from the concave surface of the bar. Find the maximum tensile stress.

189. Hooks of Circular Section. The problem of finding the maximum unit tensile stress in a hook of circular section is solved by means of Eq. (185.11). If tension is regarded as positive, the complete expression for the unit stress at the concave surface is

$$S_t = \frac{Pe(R_2 + 2\sqrt{R_1R_2} - 3R_1)}{\pi a^2 R_1(R_1 - 2\sqrt{R_1R_2} + R_2)} + \frac{P}{\pi a^2} \qquad (189.1)$$

$$S_t = \frac{P}{\pi a^2}\left[\frac{e(R_2 + 2\sqrt{R_1R_2} - 3R_1)}{R_1(R_1 - 2\sqrt{R_1R_2} + R_2)} + 1\right] \qquad (189.2)$$

At the convex surface, from Eq. (185.14),

$$S_c = \frac{P}{\pi a^2}\left[\frac{e(3R_2 - 2\sqrt{R_1R_2} - R_1)}{R_1(R_1 - 2\sqrt{R_1R_2} + R_2)} - 1\right] \qquad (189.3)$$

Example

A hook of circular section is 2 in. in diameter. The inner radius of curvature is 3 in. and the outer radius is 5 in. The load is 2,000 lb with the line of its resultant 1 in. inside the concave surface. Find the unit stress in the extreme fibers.

$$R_1 = 3 \text{ in.} \qquad R_2 = 5 \text{ in.} \qquad a = 1 \text{ in.} \qquad e = 2 \text{ in.}$$

$$\frac{P}{\pi a^2} = 636.6 \text{ psi}$$

$$S_t = 636.6\left[\frac{2(5 + 2\sqrt{15} - 9)}{3(3 - 2\sqrt{15} + 5)} + 1\right]$$

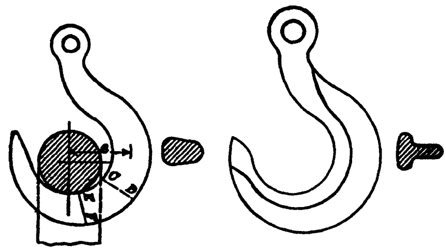
$$S_t = 636.6\left(\frac{2 \times 3.746}{3 \times 0.254} + 1\right) = 636.6 \times 10.83 = 6{,}894 \text{ psi}$$

$$S_c = 636.6\left(\frac{2 \times 4.254}{5 \times 0.254} - 1\right) = 636.6 \times 5.699 = 3{,}628 \text{ psi}$$

Problem

189-1. A hook of circular section is 3 in. in diameter. The inner radius of curvature is 4 in. and the distance from the center of the section to the load line is 3 in. If the maximum allowable unit stress is 10,000 psi, what is the safe load?

190. Hooks of Trapezoidal Section. Hooks are frequently made of trapezoidal section with the larger base toward the center of curvature. In the actual hooks the corners are rounded as in Fig. 261. Such a hook may be calculated as if it were the full trapezoidal section and the bending stress in the actual hook may be computed by multiplying the stress obtained from the full trapezoid by the ratio of the moment of inertia of the full trapezoid to the moment of inertia of the actual section.



I II

FIG. 261. Hooks.

Example

A hook of trapezoidal section is 2 in. wide at the concave surface, 1 in. wide at the convex surface, and 4 in. deep between these surfaces at the section most remote from the line of the load. The inner radius is 4 in. and the line of the load is 2 in. from the concave surface. Find the unit stress in the extreme fibers when the load is 8,000 lb.

Since the inner breadth is twice the outer breadth, the ratios of Table 21 apply for $h/R_1 = 1$.

The calculated stress at the 2-in. base as a straight beam is $S = M\,{}^{3}\!/_{13}$. Since the center of gravity of the section is ${}^{16}\!/_{9}$ in. from the base, the moment is

$$M = 8{,}000 \times {}^{34}\!/_{9}.$$

The radio from the table is 1.2828.

$$S_1 = {}^{3}\!/_{13} \times 8{,}000 \times {}^{34}\!/_{9} \times 1.2828 = 8{,}947 \text{ psi}$$

$$S_1 + \frac{P}{A} = 8{,}947 + 1{,}333 = 10{,}280 \text{ psi}$$

which is the unit stress at the inner surface.

Problems

190-1. Calculate the stress in the outer fibers for the hook of the preceding example.

190-2. Solve the preceding example if the inner width is 3 in. and all other data are unchanged.

191. Miscellaneous Problems

191-1. A curved beam has an isosceles triangular c oss section with a 3-in. base and 4-in. altitude. The center of curvature is nearest the apex and 5 in. from it. Derive the expression for the distance to the neutral axis. If the bending moment is 2,000 ft-lb, find the unit stresses in the extreme fibers and compare with straight beam stresses.

191-2. A 4- by 2-in. flat steel bar is formed into a hook with an inside radius of 1 in. An 8,000-lb load is applied coinciding with the center of curvature. Find the maximum and minimum stresses.

191-3. A 10-ton crane hook of trapezoidal section has the following dimensions: Outer base 1 in., inner base 3.5 in., altitude 4.5 in., radius of curvature to inner base 2.5 in. The load line passes through the center of curvature. Find the maximum stresses on the inner and outer surfaces.

CHAPTER 19

PROPERTIES OF AREAS

192. Center of Gravity of Some Areas. The location of the center of gravity of an area is determined by the expressions

$$\bar{x} = \frac{\int x\,dA}{A} \qquad \bar{y} = \frac{\int y\,dA}{A} \tag{192.1}$$

The center of gravity of an area is frequently called the *centroid.*

Example

Locate the center of gravity of the area which is bounded by the X axis, the ordinate $x = x_1$, and the curve $y = x^n$.

$$\text{Area} = \int_0^{x_1} x^n\,dx = \frac{\left[x^{n+1}\right]_0^{x_1}}{n+1} = \frac{x_1^{n+1}}{n+1} \tag{192.2}$$

$$\text{Moment of area} = \int_0^{x_1} x^{n+1}\,dx = \frac{\left[x^{n+2}\right]_0^{x_1}}{n+2} = \frac{x_1^{n+2}}{n+2} \tag{192.3}$$

$$\bar{x} = \frac{n+1}{n+2}x_1 \tag{192.4}$$

If $y = x^n$ and $y_1 = x_1^n$

$$\bar{y} = \frac{\int_0^{x_1} x^{2n}\,dx}{2A} = \frac{x_1^{2n+1}}{2n+1} \times \frac{2(n+1)}{x_1^{n+1}} = \frac{(n+1)x_1^n}{2(2n+1)} = \frac{n+1}{2(2n+1)}y_1 \tag{192.5}$$

Table 24 gives the center of gravity of some areas which are largely used in the application of area moments.

193. Moment of Inertia of Areas. The moment of inertia of an area is defined mathematically by the expression $\int r^2\,dA$, in which dA is an element of area and r is the distance of every part of this element from the axis of reference.

Polar moment of inertia is taken with reference to an axis which is perpendicular to the plane of the area. In rectangular coordinates, $r^2 = x^2 + y^2$ and $dA = dx\,dy$. In polar coordinates, $dA = r\,d\theta\,dr$.

The polar moment of inertia of a circle of radius a is $J = \pi a^4/2$; the square of the radius of gyration is $r^2 = a^2/2$. For a hollow circle of

outside radius a and inside radius b,

$$J = \frac{\pi(a^4 - b^4)}{2} \qquad r^2 = \frac{a^2 + b^2}{2} \tag{193.1}$$

The moment of inertia of a plane area with respect to an axis in its plane is called the *moment of inertia* of the area, without any descriptive term. The moment of inertia with respect to the X axis is

$$I_x = \int y^2 \, dA$$

The moment of inertia with respect to the Y axis is

$$I_y = \int x^2 \, dA$$

Moment of inertia of any area may be transferred from any axis through the center of gravity to a parallel axis by means of the equation

$$I = I_c + Ad^2 \tag{193.2}$$

in which I_c is the moment of inertia with respect to the axis through the center of gravitv and d is the distance from this axis to the parallel axis.

Table 23 gives the moment of inertia of a few areas which are convenient in the calculation of the properties of sections which are used in structures.

Problems

193-1. Derive the expression for the moment of inertia of a rectangle with respect to an axis through the center of gravity parallel to the base (see any textbook of mechanics for the method). By transfer of axis, find the moment of inertia with respect to the base.

193-2. Find the moment of inertia of a 6- by 8-in. rectangle with respect to an axis which is outside the rectangle at a distance of 4 in. from one 6-in. edge. Solve by transfer formula. Solve also by assuming that the rectangle is extended to the axis of inertia.

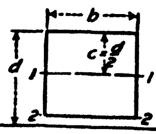
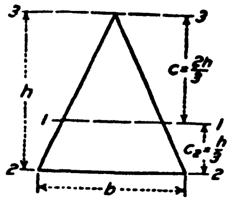
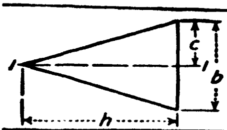
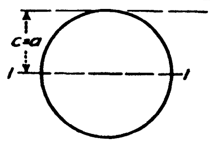
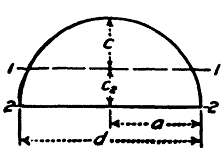
Ans. $\frac{6 \times 8^3}{12} + 48 \times 64 = 3{,}328$ in.^{4}; $\frac{6 \times 12^3}{3} - \frac{6 \times 4^3}{3} = 3{,}328$ in.^{4}

193-3. By integration, derive the expression for the moment of inertia of a triangle with respect to an axis through the vertex parallel to the base. Then transfer *to* the parallel line through the center of gravity. Finally transfer *from* the center of gravity to the base.

193-4. By integration, find the moment of inertia of a circle with respect to a diameter. Find the moment of inertia of a semicircle with respect to any diameter. *Ans.* I of semicircle $= \pi a^4/8$.

193-5. By transfer, find the moment of inertia of a semicircle with respect to a line which is parallel to the bounding diameter and passes through the center of gravity. Check with Table 23.

TABLE 23. MOMENTS OF INERTIA AND SECTION MODULI

	**Rectangle:** $A = b\,d.$ **Axis 1–1:** $I = \frac{b\,d^3}{12}$ $\quad r = \frac{d}{\sqrt{12}} = 0.288675\,d.$ $c = \frac{d}{2}$ $\quad Z = \frac{b\,d^2}{6}.$ **Axis 2–2:** $I = \frac{b\,d^3}{3}$ $\quad r = \frac{d}{\sqrt{3}} = 0.577350\,d.$
	**Any triangle:** $A = \frac{b\,h}{2}.$ **Axis 1–1:** $I = \frac{b\,h^3}{36}$ $\quad r = \frac{h}{\sqrt{18}} = 0.235702\,h.$ $c = \frac{2\,h}{3};$ $\quad Z = \frac{b\,h^2}{24}.$ $c_2 = \frac{h}{3};$ $\quad Z_2 = \frac{b\,h^2}{12}.$ **Axis 2–2:** $I = \frac{b\,h^3}{12}$ $\quad r = \frac{h}{\sqrt{6}} = 0.408248\,h.$ **Axis 3–3:** $I = \frac{b\,h^3}{4}$ $\quad r = \frac{h}{\sqrt{2}} = 0.707107\,h.$
	**Isosceles triangle:** $A = \frac{b\,h}{2}.$ **Axis 1–1:** $I = \frac{h\,b^3}{48}$ $\quad r = \frac{b}{\sqrt{24}} = 0.204124\,b.$ $c = \frac{b}{2};$ $\quad Z = \frac{h\,b^2}{24}.$
	Circle: $A = \pi a^2 = 0.785398\,d^2$ Axis 1–1: $I = \frac{\pi a^4}{4} = \frac{\pi d^4}{64} = 0.0490874\,d^4$ $r = \frac{a}{2};$ $\quad c = a = \frac{d}{2}.$ $Z = \frac{\pi a^3}{4} = \frac{\pi d^3}{32} = 0.0981748\,d^3.$
	Semicircle: $A = \frac{\pi a^2}{2} = 1.570796\,a^2.$ Axis 1–1: $I = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)a^4 = 0.109756\,a^4.$ $r = \frac{\sqrt{9\pi^2 - 64}}{6\pi}\,a = 0.264336\,a.$ $c = a - \frac{4a}{3\pi} = 0.5755868\,a;$ $\quad Z = 0.190687\,a^3.$ $c_2 = \frac{4a}{3\pi} = 0.4244132\,a;$ $\quad Z_2 = 0.258609\,a^3.$

Example 1

A trapezoid has a lower base of 16 in., an upper base of 4 in., and a height of 12 in. Find the moment of inertia with respect to the lower base; then find the moment of inertia with respect to an axis which is parallel to the base and passes through the center of gravity.

Figure 262,I shows the trapezoid divided into a parallelogram and a triangle. Figure 262,II is made up of two triangles. $\bar{y} = 4.8$ in. The moment of inertia with respect to the base is

$$\text{Parallelogram:} \quad \frac{4 \times 12 \times 12 \times 12}{3} = 2{,}304 \text{ in.}^4$$

$$\text{Triangle:} \quad \frac{12 \times 12 \times 12 \times 12}{12} = 1{,}728 \text{ in.}^4$$

$$I = \overline{4{,}032} \text{ in.}^4$$

$$I_c = 4{,}032 - 120 \times 4.8^2 = 4{,}032 - 2{,}764.8 = 1{,}267.2 \text{ in.}^4$$

Problems

193-6. Find the center of gravity and the moment of inertia with respect to the base for the trapezoid of Example 1 by means of the triangles of Fig. 262,II.

193-7. Solve Example 1 by taking the moment of inertia with respect to the upper base and then transferring to the parallel line through the center of gravity.

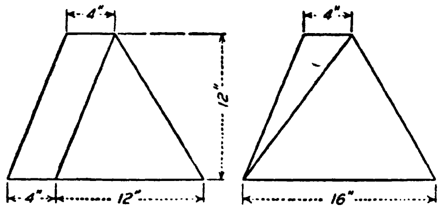


FIG. 262. Subdivision of trapezoid for computation.

193-8. Find the section modulus of the trapezoid of Fig. 262 with respect to upper and lower extreme fibers. *Ans.* $Z = 176$ in.^{3}; $Z_2 = 264$ in^3.

Example 2

Find the moment of inertia and radius of gyration of an 8- by 7-in. by 1-in. angle section with respect to an axis through the center of gravity parallel to the 7-in. leg (Fig. 263).

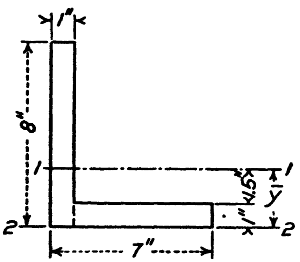


FIG. 263. Angle section.

Dividing the angle into an 8- by 1-in. rectangle and a 1- by 6-in. rectangle and taking moments about the back of the 7-in. leg,

$$\begin{aligned} 8 \times 4 &= 32 \\ 6 \times 0.5 &= 3 \\ \hline 14\bar{y} &= 35 \\ \bar{y} &= 2.5 \text{ in.} \end{aligned}$$

The moment of inertia with respect to the axis 1-1 may be found by taking the moment of inertia of each rectangle with respect to a parallel axis through its center of gravity and then transferring to axis 1-1.

$$I_c = \frac{1 \times 8^3}{12} + 8 \times \tfrac{9}{4} + \frac{6 \times 1^3}{12} + 6 \times 2^2 = 42\tfrac{2}{3} + 18 + \tfrac{1}{2} + 24 = 85\tfrac{1}{6}$$

Another method is to find the moment of inertia about some axis which is a common base for both rectangles and then transfer *to* the center of gravity. Using axis 2-2 at the back of the 7-in. leg,

$$I = \frac{1 \times 8^3}{3} + \frac{6 \times 1^3}{3} = 172\tfrac{2}{3} \qquad I_c = 172\tfrac{2}{3} - 14 \times \tfrac{25}{4} = 85\tfrac{1}{6}$$

Problems

193-9. Divide the angle section of Fig. 263 into two 7-in. rectangles; find the sum of the moments of inertia with respect to the right edge of the 7-in. leg; and transfer 1.5 in. to axis 1-1.

193-10. Find the moment of inertia of the section of Fig. 263 with respect to the axis through the center of gravity parallel to the 8-in. leg. Solve by two methods.

193-11. Compare the moments of inertia, area, and radii of gyration of the section of Fig. 263 with the values of a similar section which is given in the handbook.

193-12. Find the location of the center of gravity for a 6- by 5- by 1-in. angle section. Compute the moment of inertia for axes through the center of gravity parallel to each leg. Calculate the radius of gyration for each of these and compare all the results with the corresponding values for a 3- by 2½- by ½-in. angle section which is given in the AISC handbook.

Figure 264 shows a standard channel section. The moment of inertia with respect to the axis 1–1 is the moment of inertia of the

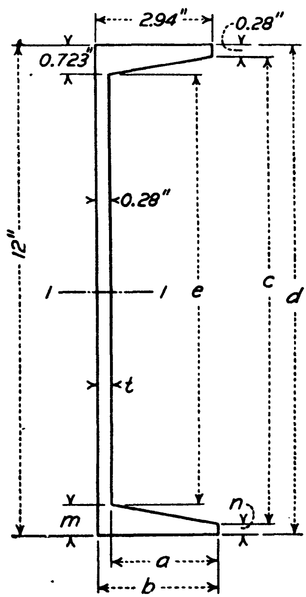


FIG. 264. Standard channel section.

rectangle of breadth b and height d minus the moment of inertia of the trapezoid which is included between the flanges. This trapezoid is a portion of an isosceles triangle which has a base c. Since the slope of the inner surfaces of the flanges in most standard channels and I beams

is 1:6, the altitude of this triangle is $3c$. The altitude of that portion of the triangle from the right face of the web to the vertex is $3e$.

$$e = c - \frac{a}{3} \tag{193.3}$$

in which a is the net width of the flange.

$$I = \frac{bd^3}{12} - \frac{3c \times c^3 - 3e \times e^3}{48} = \frac{bd^3}{12} - \frac{c^4 - e^4}{16} \tag{193.4}$$

In a similar way the moment of inertia of a standard I beam is

$$I = \frac{bd^3}{12} - \frac{c^4 - e^4}{8} \tag{193.5}$$

Example 3

Find the moment of inertia of a 12-in. 20.7-lb standard channel section with respect to the axis of symmetry.

$$c = 12 - 2 \times 0.28 = 11.44 \qquad m = 0.28 + \frac{2.66}{6} = 0.723$$

$$e = 12 - 2 \times 0.723 = 10.553$$

$$I = \frac{bd^3}{12} = \frac{2.94(12)^3 - 2.66(11.44)^3}{12} = 423.36 - 331.87$$
$$= 91.49$$

Next consider the two triangles,

$$I = \frac{bh^3}{36} = 2\,\frac{2.66(0.443)^3}{36} = 0.013$$

Transferring the triangles to the axis of symmetry of the channel by the second term of Eq. (193.2),

$$Ad^2 = 2\,\frac{2.66(0.443)}{2}(5.572)^2 = \underline{36.59}$$
$$I = 128.09 \text{ in.}^4$$

The AISC Manual of Steel Construction gives 128.1. The problem may also be solved by the use of the Eq. (193.4).

A comparison of the thickness of web and width of flange for the other 12-inch standard channels will show that the heavier sections are made by increasing the thickness of the web and the width of the flange by the same amount. The net dimensions of the flange are unchanged. A similar situation occurs in standard I beams.

Problems

193-13. Look up in the tables of the handbook the additional thickness of the 30-lb 12-in. channel. From the answer of Example 3 and the moment of

inertia of the added rectangle, calculate the moment of inertia of this channel with respect to the axis of symmetry.

193-14. Solve Prob. 193-13 for the 12-in. 40-lb channel.

193-15. Find the moment of inertia of a 20-in. 85-lb standard I beam with respect to the axis of symmetry which is perpendicular to the web.

Ans. $c = 20 - 1.30 = 18.7 \qquad e = 18.7 - (3.2/3) = 52.9/3$

$$\overline{I} = \frac{14{,}106}{3} - \frac{122{,}283.1 - 96{,}680.6}{8} = 4{,}702 - 3{,}200.4$$

$$I = 1{,}501.6$$

Example 4

Find the moment of inertia of a 12-in. 25-lb standard channel with respect to the axis through the center of gravity which is parallel to the web.

The moment of inertia is calculated first with respect to the right edge of the web. This is a common base for the three rectangles and two triangles which make up the section.

$$I = \frac{12 \times 0.387^3}{3} + \frac{0.56 \times 2.66^3}{3} + \frac{0.887 \times 2.66^3}{12} = 5.1362 \text{ in.}^4$$

$$7.314\bar{x} = 2.1298 \qquad \bar{x} = 0.2912 \text{ in.}$$

$$I_c = 5.1362 - 7.314 \times 0.291^2 = 4.52 \text{ in.}^4$$

Problems

193-16. Solve Example 4 for a 12-in. 20.7-lb standard channel. Make use of part of the work of the example.

193-17. Find the moment of inertia of a 20-in. 95-lb standard I beam with respect to the axis parallel to the web through its center of gravity. Find the moment of inertia of the one large rectangle and the two smaller rectangles about the required axis. Find the moment of inertia of one of the four equal triangles with respect to a parallel axis through its center of gravity, and then transfer.

193-18. Find the moment of inertia of the section of Fig. 265 with respect to the axis of symmetry and also with respect to the axis through the center of

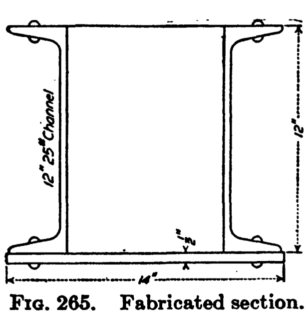


FIG. 265. Fabricated section.

gravity parallel to the plate. Get as many of the data from the handbook as possible. This column is latticed at the top of the figure. The latticing is not considered in the calculations.

The sum of the moments of inertia of a plane area with respect to a pair of axes at right angles to each other is equal to the polar moment of inertia of the area with respect to the axis which is perpendicular to this plane and passes through the intersection of the two axes which lie in the plane.

If one of these axes in the plane is the X axis,

$$I_x = \int y^2\,dA \tag{193.6}$$

If the other axis at right angles is the Y axis,

$$I_y = \int x^2\,dA \tag{193.7}$$

For the polar moment of inertia, $r^2 = x^2 + y^2$.

$$J = \int (x^2 + y^2)\,dA = \int x^2\,dA + \int y^2\,dA \tag{193.8}$$

For angle sections with unequal legs, the handbooks give the moments of inertia with respect to axes parallel to each leg and also give the minimum radius of gyration. The maximum moment of inertia may be found from the relation

$$I_x + I_y = J = I_{\min} + I_{\max} \tag{193.9}$$

Problems

193-19. Find the maximum moment of inertia for a 4- by 3- by ½-in. angle section for an axis through the center of gravity. Get I_x and I_y from the handbook. From the minimum radius of gyration, which is given, calculate the minimum moment of inertia, then solve for $I_{\max}$.

193-20. Solve Prob. 193-19 for a 6- by 3½- by ⅜-in. zee.

194. Change of Direction of Axes for Moment of Inertia. By means of Eq. (193.2), moment of inertia may be transferred from one axis to a parallel axis, provided one of these axes passes through the center of gravity. It is frequently necessary to find the moment of inertia with respect to an axis which is inclined to the principal lines of the figure in such a way that the solution by direct integration is difficult. If the moment of inertia of an area is known for any two axes in the plane at right angles to each other, the moment of inertia for any other axis at a known angle with those axes may be calculated.

Figure 266 represents any area in the XY plane. The moment of inertia of this area with respect to the X axis may be designated by I_x and the moment of inertia with respect to the Y axis may be designated by I_y.

$$I_x = \int y^2\, dA$$
$$I_y = \int x^2\, dA$$

The line OX' is a new axis which makes an angle θ with the X axis and OY' is an axis at right angles to OX'. The coordinates of an element of area dA with respect to these new axes are (x', y').

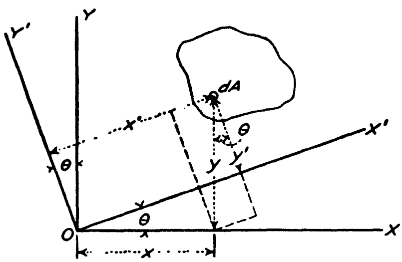


FIG. 266. Change of direction of axes.

The moment of inertia of the area with respect to OX' is

$$I = \int y'^2\, dA \tag{194.1}$$

From the geometry of the figure,

$$y' = y \cos \theta - x \sin \theta \tag{194.2}$$

$$I = \int (y^2 \cos^2 \theta - 2xy \cos \theta \sin \theta + x^2 \sin^2 \theta)\, dA \tag{194.3}$$

$$I = I_x \cos^2 \theta + I_y \sin^2 \theta - 2 \cos \theta \sin \theta \int xy\, dA \tag{194.4}$$

$$I = I_x \cos^2 \theta + I_y \sin^2 \theta - \sin 2\theta \int xy\, dA \tag{194.5}$$

$$I = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - \sin 2\theta \int xy\, dA \tag{194.6}$$

195. Product of Inertia. The expression $\int xy\, dA$, which occurs in the above equations; is called the *product of inertia.* The product of inertia has no physical meaning but it is a convenient tool in finding the moment of inertia of a plane area with respect to any axis.

If an area is symmetrical with respect to either one of a pair of rectangular axes, its product of inertia with respect to that pair of axes is zero. Figure 267 represents an area which is symmetrical with respect to the Y axis. If this is integrated first with respect to x,

$$H = \tfrac{1}{2} \int [x^2]_{x_1}^{x_2} y\, dy = \tfrac{1}{2} \int (x_2^2 - x_1^2) y\, dy$$

If the area is symmetrical with respect to the Y axis, the lower limit x_1 is numerically equal and opposite in sign to the upper limit x_2, and the squares are the same in magnitude and sign; consequently the term in the brackets vanishes and

$$H = 0$$

When the product of inertia is known with respect to a pair of rectangular axes through the center of gravity of an area, it may be calculated for a second pair of parallel axes in the plane of the area by a formula which is similar to the transfer equation for moment of inertia.

Let OX, OY (Fig. 268) be the original pair of axes through the center of gravity, and let (x, y) be the coordinates of an element dA with reference to these axes. Let $O'X'$, $O'Y'$ be a new pair of parallel axes.

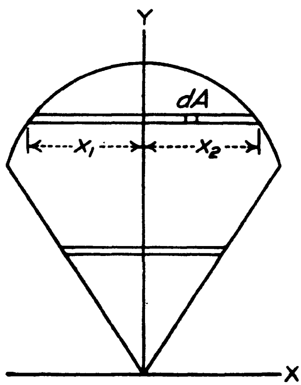


FIG. 267. Symmetrical section.

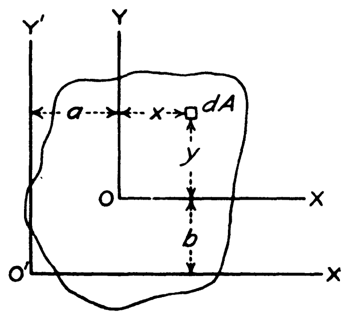


FIG. 268. Transfer of product of inertia.

Let (a, b) be the coordinates of the center of gravity of the area with respect to the new axes.

If H is the product of inertia with respect to the new axes,

$$H = \int (a + x)(b + y)\, dA \tag{195.1}$$

$$H = ab \int dA + b \int x\, dA + a \int y\, dA + \int xy\, dA \tag{195.2}$$

$$H = abA + 0 + 0 + H_c \tag{195.3}$$

where H_c is the product of inertia with respect to the axes through the center of gravity.

If the center of gravity falls in the first or third quadrant with respect to the axes for which the product of inertia is desired, the product abA is positive, and H will be positive unless H_c is negative and numerically

greater than abA. If the center of gravity falls in the second quadrant, abA is negative since a is negative; if it falls in the fourth quadrant, abA is negative because b is negative.

Problems

195-1. By integration, find the product of inertia of a rectangle of base b and altitude d with respect to the lower and left edges as axes.

Ans. $H = b^2d^2/4$.

195-2. Solve Prob. 195-1 with respect to the lower and right edges as axes.

Ans. $H = -b^2d^2/4$.

Example

Find the product of inertia of the semicircle of Fig. 269 with respect to the diameter and the tangent at the bottom. By rectangular coordinates with the origin at the center of the circle,

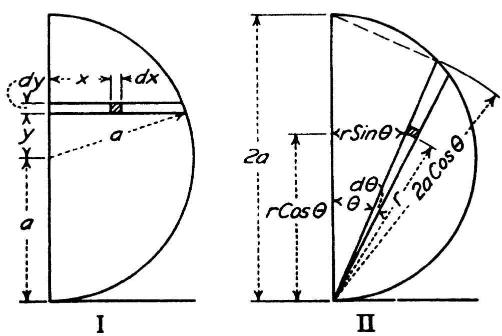


FIG. 269. Product of inertia of semicircle.

$$H = \int\int (a + y)x\,dx\,dy = \int \frac{a+y}{2}\Big[x^2\Big]_0^{a^2-y^2} dx \tag{195.4}$$

$$H = \frac{1}{2}\int (a+y)(a^2 - y^2)\,dy = \frac{1}{2}\left[a^3y + \frac{a^2y^2}{2} - \frac{ay^3}{3} - \frac{y^4}{4}\right]_{-a}^{a}$$

$$H = \frac{2a^4}{3}$$

By polar coordinates with the origin at the tangent point,

$$H = \int\int \sin\theta\cos\theta r^3\,dr\,d\theta = \int \sin\theta\cos\theta\left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$H = 4a^4\int \cos^5\theta\sin\theta\,d\theta = -\frac{2a^4}{3}\Big[\cos^6\theta\Big]_0^{\pi/2} = \frac{2a^4}{3}$$

By transfer,

$$a = \frac{4a}{3\pi} \qquad b = a \qquad A = \frac{\pi a^2}{2}$$

$$H = 0 + \frac{4a \times a \times \pi a^2}{2 \times 3 \times \pi} = \frac{2a^4}{3}$$

Problems

195-3. By integration, find the product of inertia for the right-angled triangle of Fig. 270 with respect to the base and the left edge as axes. Integrate first with respect to y.

Ans. The first integration gives $H = h^2/2b^2 \int_0^b (b - x)^2 x\, dx$; $H = h^2b^2/24$.

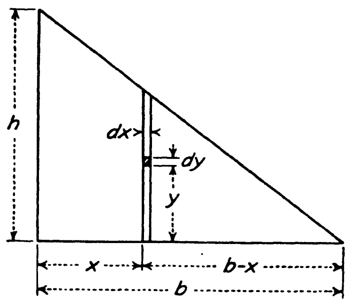


FIG. 270. Product of inertia of right-angled triangle.

195-4. By transfer, find the product of inertia of the triangle of Fig. 270 with respect to the parallel axes through the center of gravity.

Ans. $H_c = -b^2h^2/72$.

195-5. Find the product of inertia of a 6- by 5- by 1-in. angle section with respect to axes through the center of gravity parallel to the legs. The section may

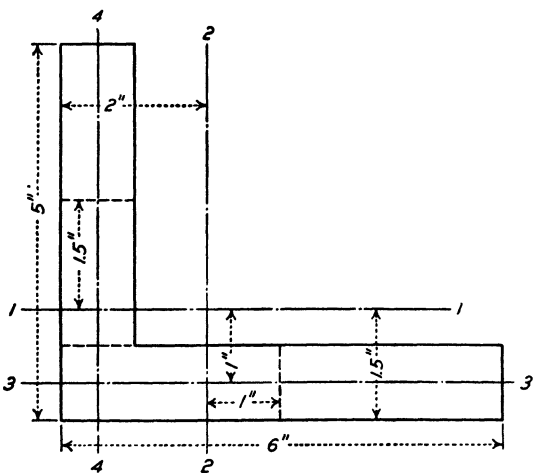


FIG. 271. Product of inertia of angle section.

be divided into two rectangles. The product of inertia of each of these with respect to the axes through their center of gravity is zero. Transfer to the axes 1–1, 2–2 and add.

$H_c = 4 \times (-1.5) \times 1.5 + 0 + 6 \times 1 \times (-1) + 0 = -9 - 6 = -15\text{ in.}^4$

The problem may be solved more readily in another way. Since 3–3 is an axis of symmetry for the horizontal leg, the product of inertia of this rectangle for 3–3 and any other axis (as 4–4) is zero. Since axis 4–4 is a line of symmetry for the vertical leg, the product of inertia of this rectangle for axes 4–4 and 3–3 is zero. The product of inertia for the entire section for axes 3–3 and 4–4 is the sum of products of inertia of the separate rectangles; therefore, $H = 0$. When the product of inertia is transferred from 3–3, 4–4 to axes 1–1, 2–2, the equation is

$$0 = H_c + 10 \times 1.5 \times 1 \qquad H_c = -15 \text{ in.}^4$$

196. Calculation of Moment of Inertia. For the calculation of moment of inertia, Eq. (194.6) becomes

$$I = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - H \sin 2\theta \tag{196.1}$$

Equation (194.5) becomes

$$I = I_x \cos^2 \theta + I_y \sin^2 \theta - 2H \sin \theta \cos \theta \tag{196.2}$$

Equation (196.1) is generally used. However, when $\sin \theta$ and $\cos \theta$ are fractions which may be conveniently squared, Eq. (196.2) is preferable.

Problems

196-1. Find the moment of inertia of a 4- by 3-in. rectangle (Fig. 272) with respect to an axis through the lower left corner making an angle of 20° upward from the 4-in. edge.

Ans. $I_x = 36 \text{ in.}^4 \qquad I_y = 64 \text{ in.}^4 \qquad H = 36 \text{ in.}^4$

$$I = \frac{36 + 64}{2} + \frac{36 - 64}{2} \cos 40° - 36 \sin 40°$$

$$I = 50 - 14 \times 0.7660 - 36 \times 0.6428 = 16.14 \text{ in.}^4$$

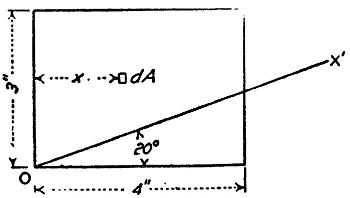


FIG. 272. Change of direction of axis.

196-2. Solve Prob. 196-1 if the axis is 20° below the direction of the 4-in. edge.

Ans. $I = 62.42 \text{ in.}^4$

196-3. Solve Probs. 196-1 and 196-2 if the axes are $\tan^{-1} \frac{3}{4}$ and $\tan^{-1} (-\frac{4}{3})$. Use Eq. (196.2) without trigonometric tables. Check one of these by means of the moment of inertia of the two equal triangles with respect to the diagonal as the common base.

196-4. Find the moment of inertia of the 4- by 4- by ½-in. angle section of Fig. 273 with respect to the axis 3–3 by means of Eq. (196.1). Take I_x from the handbook.

196-5. Find the moment of inertia of the section of Fig. 273 with respect to the line which bisects the angle. Check by subtracting the moment of inertia of a 3-in. square with respect to its diagonal from the moment of inertia of a 4-in. square.

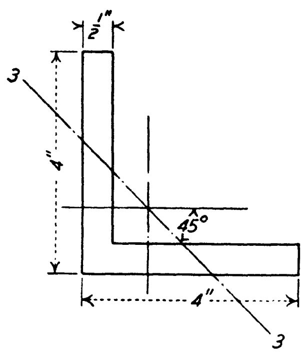


FIG. 273. A principal axis of angle section.

196-6. Check the answers of Probs. 196-4 and 196-5 by means of Eq. (193.9).

197. Change of Direction of Axes for Product of Inertia. To derive the expression for the product of inertia for the axes OX', OY' of Fig. 266, the fundamental integral is

$$H' = \int x'y'\,dA \tag{197.1}$$

$$H' = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\,dA \tag{197.2}$$

$$H' = (\cos^2\theta - \sin^2\theta)\int xy\,dA + \cos\theta\sin\theta\int (y^2 - x^2)\,dA \tag{197.3}$$

$$H' = H\cos 2\theta + \frac{I_x - I_y}{2}\sin 2\theta \tag{197.4}$$

When the expression to the right of the equality sign in Eq. (197.4) is made equal to zero,

$$\tan 2\theta = \frac{2H}{I_y - I_x} \tag{197.5}$$

This gives the angle at which the product of inertia is zero.

Problems

197-1. In the 4- by 3-in. rectangle of Fig. 272, what will be the angle between OX' and the 4-in. edge if the product of inertia with respect to OX' and the axis through O normal to it is zero? *Ans.* $\theta = 34°32'$.

197-2. Find the direction of the pair of axes through the center of gravity of the 6- by 5- by 1-in. angle section of Fig. 271 for which the product of inertia is zero.

198. Direction of Axis for Maximum Moment of Inertia. From Eq. (194.6) the moment of inertia with respect to an axis at an angle θ with the X axis is

$$I = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - H \sin 2\theta \tag{198.1}$$

The direction of the axis for maximum or minimum moment of inertia is found by differentiating Eq. (198.1) with respect to θ and equating the derivative to zero.

$$\frac{dI}{d\theta} = (I_y - I_x) \sin 2\theta - 2H \cos 2\theta \tag{198.2}$$

from which the condition of maximum or minimum is

$$\tan 2\theta = \frac{2H}{I_y - I_x} \tag{198.3}$$

A comparison of Eq. (198.3) with Eq. (197.5) shows that the condition for maximum and minimum moment of inertia is identical with the condition for zero product of inertia. There are two solutions for Eq. (198.3), which give values of 2θ differing by 180° and values of θ differing by 90°. One of these gives the direction of the axis for which the moment of inertia is a maximum and the other gives the direction of the axis for which the moment of inertia is a minimum.

Since the product of inertia for an axis of symmetry is zero, the moment of inertia with respect to an axis of symmetry is greater or less than the moment of inertia for any other axis through any given point in its line.

The line which bisects the angle between the legs of an angle section of equal legs is a line of symmetry and the moment of inertia for this axis is greater than for any other axis through the center of gravity, while the moment of inertia for the axis at right angles to this line of symmetry (the axis 3–3 of Fig. 273) is smaller than that for any other axis through the center of gravity.

The maximum and minimum moments of inertia for axes through the center of gravity are called the *principal moments of inertia* and the axes are the *principal axes* of the area. After the moment of inertia and the product of inertia have been found with respect to any convenient pair of axes which are perpendicular to each other in the plane

of the area, the direction of the principal axes is found by Eq. (198.3) or from the condition that an axis of symmetry is a principal axis. The angle thus found is then substituted in Eq. (198.1) or (198.2) to find the required principal moments of inertia.

Example

Find the principal moments of inertia for the 6- by 5- by 1-in. angle section for axes through the center of gravity.

$$\tan 2\theta = \frac{-30}{(200/6) - (125/6)} = -2.4 \qquad 2\theta = 112°37'.2 \text{ or } 292°37'.2.$$

From the first angle, $\cos 2\theta = -5/13$ and $\sin 2\theta = 12/13$:

$$I_m = 325/12 - 75/12(-5/13) - (-15)12/13 = 130/3 = I_{\max}$$

After the calculation has been completed, the result is found to be the maximum, since it is larger than I_x or I_y.

$$I_{\min} = 325/12 - 75/12 \times 5/13 - (-15)(-12/13) = 325/12 - 195/12 = 65/6$$

When the sine of θ and the cosine of θ may be squared conveniently, Eq. (196.2) may be used. From $\tan 2\theta = -2.4$, $\tan \theta$ is found to be $-2/3$, for which θ is $-33°41'$ or $\tan \theta = 3/2$, for which $\theta = 56°19'$.

Problems

198-1. Solve the foregoing example for maximum I from $\tan \theta = 3/2$. Calculate the sine and cosine without the use of tables.

198-2. Solve the example for $I_{\min}$ by means of the single angle.

198-3. Find the moment of inertia for the angle section of the foregoing example at 45° and at 60°. Compare with $I_{\max}$.

198-4. Find the maximum and minimum moments of inertia of a 4- by 3-in. rectangle with respect to axes through the lower left corner.

Ans. $I_{\min} = 11.374$ in.^{4} at 34°22'.5; $I_{\max} = 88.626$ in^4.

198-5. Find the principal moments of inertia of the right-angled triangle of Fig. 274.

Ans. $I_{\max} = 525.05$ in.^{4} with respect to an axis at 60°08' with the horizontal toward the right;

$I_{\min} = 337.500 - 47.616 - 139.932 = 149.95$ in.^{4}

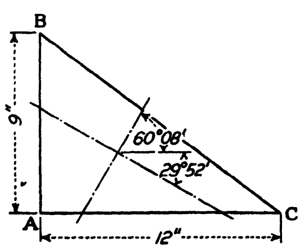


FIG. 274. Principal axes of a right-angled triangle.

198-6. A cantilever 10 ft long has a solid triangular section equivalent to Fig. 274 which is rotated 29°52′ to bring the principal axis of minimum moment of inertia into the horizontal. Find the stress of each corner when a load of 300 lb is placed on the free end.

Ans. The perpendicular distance from the corner A to the principal axis is

$$c = 3 \cos 29°52' + 4 \sin 29°52' = 3 \times 0.86719 + 4 \times 0.49798$$

$$c = 4.5935 \qquad s = \frac{36{,}000 \times 4.5935}{149.95} = 1{,}103 \text{ psi compression}$$

$$\text{At } B,\ s = \frac{36{,}000 \times 3.2112}{149.95} = 771 \text{ psi tension.}$$

$$\text{At } C,\ s = \frac{36{,}000 \times 1.3823}{149.95} = 332 \text{ psi tension.}$$

199. Calculation by Means of the Tangent. Equation (196.1) or (196.2) may be used to calculate the moment of inertia at any angle and to find the minimum moments of inertia after the angle has been determined by Eq. (197.5). Another form, which is applicable to transfer to or from maximum or minimum moments of inertia only, makes use of the tangent of the single angle, $\phi = \frac{1}{2} \tan^{-1} [2H/(I_y - I_x)]$.

By using ϕ and 2ϕ instead of θ and 2θ in the previous equations in order to express the limitations of the equations which follow,

$$\tan 2\phi = \frac{\sin 2\phi}{\cos 2\phi} = \frac{2H}{I_y - I_x} \tag{199.1}$$

$$I_y \sin \phi \cos \phi - I_x \sin \phi \cos \phi - H(\cos^2 \phi - \sin^2 \phi) = 0 \tag{199.2}$$

$$I_y \sin^2 \phi - I_x \sin^2 \phi - H \sin \phi \cos \phi + H \frac{\sin^3 \phi}{\cos \phi} = 0 \tag{199.3}$$

From Eq. (196.2),

$$I_m = I_y \sin^2 \phi + I_x \cos^2 \phi - 2H \sin \phi \cos \phi \tag{199.4}$$

Subtracting Eq. (199.3) from Eq. (199.4),

$$I_m = I_x - H \sin \phi \frac{\sin^2 \phi + \cos^2 \phi}{\cos \phi} = I_x - H \tan \phi \tag{199.5}$$

If the position of the second axis at 90° from the first is represented by ϕ',

$$\tan 2\phi' = \frac{\sin 2\phi'}{\cos 2\phi'} = \frac{2H}{I_y - I_x} = -\frac{2 \sin \phi \cos \phi}{\sin^2 \phi - \cos^2 \phi}$$

Since $\sin \phi' = \sin [(\pi/2) + \phi] = \cos \phi$, and $\cos \phi' = -\sin \phi$,

$$-I_y \sin \phi \cos \phi + I_x \sin \phi \cos \phi - H \sin^2 \phi + H \cos^2 \phi = 0 \tag{199.6}$$

Multiplying by $\sin\phi/\cos\phi$,

$$I_x \sin^2\phi - I_y \sin^2\phi - H\frac{\sin^3\phi}{\cos\phi} + H\sin\phi\cos\phi = 0 \qquad (199.7)$$

$$I'_m = I_x\cos^2\phi' + I_y\sin^2\phi' - 2H\sin\phi'\cos\phi' \qquad (199.8)$$

$$I'_m = I_x\sin^2\phi + I_y\cos^2\phi + 2H\sin\phi\cos\phi \qquad (199.9)$$

Subtracting Eq. (199.7) from Eq. (199.9),

$$I'_m = I_y(\sin^2\phi + \cos^2\phi) + H\sin\phi\cos\phi + H\frac{\sin^3\phi}{\cos\phi} \qquad (199.10)$$

$$I'_m = I_y + H\tan\phi \qquad (199.11)$$

Example

Find the principal moments of inertia of the 6- by 5- by 1-in. angle section by Eqs. (199.5) and (199.11)

$$\tan\phi = -2/3 \qquad H = -15 \qquad H\tan\phi = 10 \qquad I_{\min} = 125/6 - 10 = 65/6$$

$$I_{\max} = 100/3 + 10 = 130/3$$

$$H\tan\phi' = (-15)(3/2) = -45/2 \qquad I_{\min} = 200/6 - 45/2 = 65/6$$

$$I_{\max} = 125/6 + 45/2 = 130/3.$$

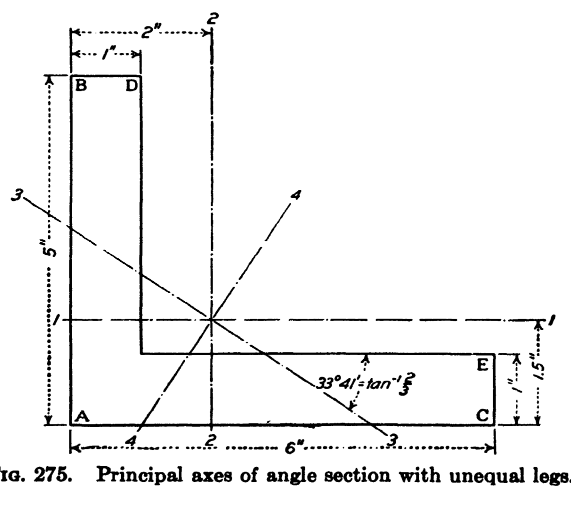


FIG. 275. Principal axes of angle section with unequal legs.

Given ϕ or ϕ', there are two values of the tangent, which are numerically the tangent and cotangent of either angle. The smaller product $H\tan\phi$ or $H\tan\phi'$ is added to the larger I_x or I_y and subtracted from the smaller I_x or I_y to get $I_{\max}$ and $I_{\min}$; or the larger product is added to the smaller I_x or I_y and subtracted from the larger. No attention need be paid to the sign of H or of the tangents.

Problems

199-1. Calculate the principal moments of inertia for the right-angled triangle of Fig. 274 by Eqs. (199.5) and (199.11) as described in foregoing statement.

199-2. Find the maximum and minimum moments of inertia for the 4- by 3-in. rectangle of Fig. 272 for axes through the lower left corner.

200. Stress and Deflection with Principal Axes Inclined. When the bending moment is not in the plane of a principal axis of inertia, it is resolved into components perpendicular to the principal axes, and the effects of these components are added to get the total stress and deflection. This method was used in Arts. 120 and 121 without proof.

Figure 276 represents any section of a beam for which XX and YY are the principal axes of inertia. The line MM, at an angle α with

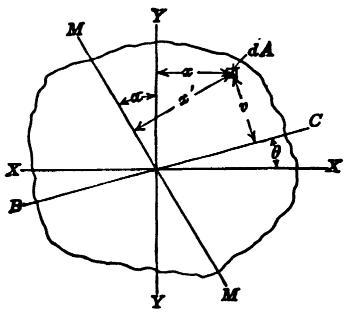


FIG. 276. Moment at angle with principal axis.

YY, is in the plane of the bending moment, and the line BC, at an angle θ with XX, is the neutral axis. The element dA, whose coordinates with respect to the *principal axes are* x and y and whose distance from the *neutral axis* is v, is subjected to a stress which varies as v.

$$v = y \cos \theta - x \sin \theta \tag{200.1}$$

$$s = kv = k(y \cos \theta - x \sin \theta) \tag{200.2}$$

Since the external moment is in the plane MM, the resisting moment must lie in the same plane; therefore, the sum of the moments of all the stress on the entire area about the line MM must be zero. The perpendicular distance from dA to the line MM is

$$x' = y \sin \alpha + x \cos \alpha \tag{200.3}$$

$$\int sx' \, dA = 0 = k \int vx' \, dA \tag{200.4}$$

$$\int y^2 \cos\theta \sin\alpha\, dA + \int xy \cos\theta \cos\alpha\, dA - \int xy \sin\theta \sin\alpha\, dA - \int x^2 \sin\theta \cos\alpha\, dA = 0 \quad (200.5)$$

$$I_x \cos\theta \sin\alpha - I_y \sin\theta \cos\alpha = 0 \quad (200.6)$$

in which I_x is the moment of inertia with respect to the axis XX, and I_y is the moment of inertia with respect to YY. The second and third terms of Eq. (200.5) include the product of inertia with respect to the principal axes, which is zero.

$$\tan\theta = \frac{I_x}{I_y}\tan\alpha$$

Example 1

A 6- by 8-in. beam is subjected to a load perpendicular to its length making an angle of 30° with the planes of the 8-in. faces. Find the angle between the neutral axis and the planes of the 6-in. faces.

When the line through the center parallel to the 6-in. faces is taken as the X axis,

$$I_x = \frac{6 \times 8^3}{12} = 256$$

$$I_y = \frac{8 \times 6^3}{12} = 144$$

$$\tan\theta = {}^{256}\!/_{144} \times 0.5774 = 1.0264$$

$$\theta = 45°45'$$

The neutral axis makes an angle of 15°45′ with the line normal to the bending moment.

From Fig. 276, the component of the bending moment perpendicular to the neutral axis is $M \cos(\theta - \alpha)$. The moment of inertia with respect to the neutral axis is $I_x \cos^2\theta + I_y \sin^2\theta$, and

$$v = y\cos\theta - x\sin\theta$$

$$s = \frac{M\cos(\theta - \alpha)v}{I_x\cos^2\theta + I_y\sin^2\theta} \quad (200.7)$$

$$s = \frac{M(\cos\theta\cos\alpha + \sin\theta\sin\alpha)(y\cos\theta - x\sin\theta)}{I_x\cos^2\theta + I_y\sin^2\theta} \quad (200.8)$$

$$s = \frac{My(\cos^2\theta\cos\alpha + \cos\theta\sin\theta\sin\alpha) - Mx(\cos\theta\sin\theta\cos\alpha + \sin^2\theta\sin\alpha)}{I_x\cos^2\theta + I_y\sin^2\theta} \quad (200.9)$$

$$My(\cos^2\theta\cos\alpha + \cos\theta\sin\theta\sin\alpha) = My\cos\alpha(\cos^2\theta + \cos\theta\sin\theta\tan\alpha) = My\cos\alpha\left(\cos^2\theta + \frac{I_y}{I_x}\sin^2\theta\right) \quad (200.10)$$

$$\frac{My(\cos^2\theta\cos\alpha + \cos\theta\sin\theta\sin\alpha)}{I_x\cos^2\theta + I_y\sin^2\theta} = \frac{My\cos\alpha}{I_x} \quad (200.11)$$

In a similar way the second part of Eq. (200.3) may be shown to be

$$\frac{Mx \sin \alpha}{I_y}$$

and

$$s = \frac{My \cos \alpha}{I_x} - \frac{Mx \sin \alpha}{I_y} \tag{200.12}$$

To find the fiber stress at any point in a beam when the bending moment is inclined to the principal axes of inertia, resolve the bending moment (or the applied forces) perpendicular to the two axes and compute the stress for each component separately. The actual unit stress is the sum of the results of these two, taken with the proper sign.

Example 2

The 6- by 5- by 1-in. angle section of Fig. 275 is used as a cantilever 5 ft long to carry a load of 500 lb on the free end. The 6-in. leg is horizontal. Find the unit stress at corner A.

Since I_x and I_y represent the principal moments of inertia in Eq. (200.12), it is convenient to draw a sketch with the 6-in. leg at 33°41′, which makes the axis of minimum moment of inertia lie in the usual horizontal X axis.

$$I_x = \frac{65}{6} \qquad I_y = \frac{130}{3} \qquad \tan \alpha = \frac{2}{3} \qquad \sin \alpha = \frac{2}{\sqrt{13}} \qquad \cos \alpha = \frac{3}{\sqrt{13}}$$

$$\frac{M \cos \alpha}{I_x} = \frac{30{,}000 \times 0.83205}{65/6} = 2{,}304 \qquad \frac{M \sin \alpha}{I_y} = \frac{30{,}000 \times 0.55470}{130/3} = 384$$

$$\begin{aligned} 1.5 \times 0.43205 &= 1.24807 \\ 2.0 \times 0.55470 &= 1.10940 \\ y &= 2.35747 \end{aligned} \qquad \begin{aligned} 2.0 \times 0.83205 &= 1.66410 \\ 1.5 \times 0.55470 &= -0.83205 \\ x &= 0.83205 \end{aligned}$$

$$\begin{aligned} 2{,}304 \times 2.35747 &= 5{,}431.6 \text{ psi compression} \\ 384 \times 0.83205 &= 319.5 \text{ psi tension} \\ S_a &= 5{,}112 \text{ psi compression} \end{aligned}$$

Problems

200-1. Solve the foregoing example for the stress at B.

Ans. $y = \dfrac{3.5 \times 3}{\sqrt{13}} - \dfrac{2 \times 2}{\sqrt{13}} = \dfrac{6.5}{\sqrt{13}} = 3.60555 \times 0.5 = 1.8028$

$x = \dfrac{7 + 6}{\sqrt{13}} = 3.6056 \qquad S_b = 4{,}154 + 1{,}384 = 5{,}538$ psi tension

200-2. Solve Example 2 for the unit stress at C and D.

Example 3

A short post has the right-angled triangular section of Fig. 274. The compression of 10,800 lb parallel to the length is applied along a line which is 2 in. from the 12-in. face and 2 in. from the 9-in. face.

From the example of the preceding article,

$$I_{\min} = I_x = 149.95 \text{ at } 29°52' \text{ with the 12-in. face}$$
$$I_{\max} = I_y = 525.05$$

The moment arm of the load with respect to the axis of I_x is

$$1 \times 0.86719 + 2 \times 0.49798 = 1.86315.$$

$$\text{Stress from this moment} = \frac{10{,}800 \times 1.86315y}{149.95} = 134.19y.$$

The moment arm of the load with respect to the axis of I_y is

$$2 \times 0.86719 - 1 \times 0.49798 = 1.2364 \text{ in.}$$

$$\text{Stress from this moment is } \frac{10{,}800 \times 1.2364x}{525.05} = 25.43x.$$

For the corner A,

$$\begin{aligned} y &= 4 \times 0.49798 + 3 \times 0.86719 = 4.5935 \\ 134.19y &= 616.4 \text{ psi compression} \\ x &= 4 \times 0.86719 - 3 \times 0.49798 = 1.9748 \\ 25.43x &= 50.2 \text{ psi compression} \\ \text{Total stress at } a &= 200 + 616.4 + 50.2 = 867 \text{ psi compression} \end{aligned}$$

Problems

200-3. Find the unit stress at B of Fig. 274 for Example 3.
200-4. Find the unit stress at C of Fig. 274 for Example 3.

201. Moment of Inertia of Regular Polygons. *The moment of inertia of a regular polygon with respect to any axis through the center of gravity is a constant.*

A regular polygon has as many axes of symmetry as it has sides. For every axis of symmetry there is a pair of axes for which the product of inertia is zero. When I_x and I_y are taken with respect to a pair of axes one of which is an axis of symmetry, Eq. (196.1) becomes

$$I = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta \tag{201.1}$$

Cosine of 2θ decreases continuously from 1 to -1 as the angle 2θ increases from 0 to π and the angle θ increases from 0 to $\pi/2$. If $I_x - I_y$ is positive, I_x is the maximum and I_y is the minimum. If $I_x - I_y$ is negative, I_x is the minimum and I_y is the maximum. Since the cosine decreases continuously, there can be no maximum or minimum between I_x and I_y. However, every regular polygon has one or more axes of symmetry in this quadrant for which the moment of inertia must be a maximum. Consequently, $I_x - I_y = 0$, and the moment of inertia is the same for all axes.

Problems

201-1. Find the moment of inertia of an equilateral triangle of side b with respect to an axis through the center of gravity parallel to one side. Also find the moment of inertia with respect to an axis of symmetry.

$$Ans.\ I_x = \frac{b^4(\sqrt{3}/2)^3}{36} = \frac{b^4\sqrt{3}}{96};\ I_y = \frac{2(b\sqrt{3}/2)(b/2)^3}{12} = \frac{b^4\sqrt{3}}{96}$$

201-2. Find the moment of inertia of a square with respect to an axis through the center of gravity parallel to a side and also with respect to a diagonal.

Since the moment of inertia of a regular polygon is the same for every axis through the center of gravity, it is equal to one-half the polar moment of inertia.

$$I = \frac{J}{2} \tag{201.2}$$

To find the polar moment of inertia, the regular polygon of n sides is divided into n isosceles triangles, one of which is shown in Fig. 277.

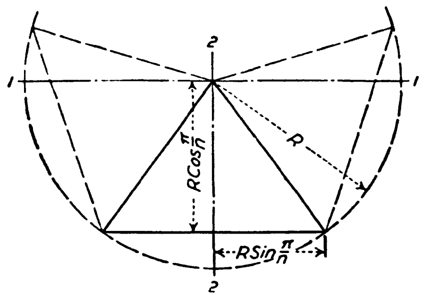


FIG. 277. Regular polygon.

This triangle has equal sides of length R, which is the radius of the circumscribed circle, and a base of $2R \sin (\pi/n)$. The altitude is $R \cos (\pi/n)$. The polar moment of inertia of the single triangle with respect to the vertex is

$$J = \frac{bh^3}{4} + \frac{b^3h}{48} = R^4\left[\frac{\cos^2 (\pi/n)}{2} + \frac{\sin^2 (\pi/n)}{6}\right] \sin\frac{\pi}{n} \cos\frac{\pi}{n} \tag{201.3}$$

For the entire polygon of n triangles,

$$I = \frac{nR^4}{8}\left(\cos^2\frac{\pi}{n} + \frac{1}{3}\sin^2\frac{\pi}{n}\right) \sin\frac{2\pi}{n} \tag{201.4}$$

The minimum section modulus is I/R.

$$Z_{\min} = \frac{nR^3}{8}\left(\cos^2\frac{\pi}{n} + \frac{1}{3}\sin^2\frac{\pi}{n}\right) \sin\frac{2\pi}{n} \tag{201.5}$$

The maximum section modulus is I/h.

$$Z_{max} = \frac{nR^3}{4}\left(\cos^2\frac{\pi}{n} + \frac{1}{3}\sin^2\frac{\pi}{n}\right)\sin\frac{\pi}{n} \tag{201.6}$$

Problems

201-3. Find the moment of inertia of a square by Eq. (201.4).

$$Ans.\ n = 4 \qquad \cos\frac{\pi}{n} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{n} \qquad R = \frac{b}{\sqrt{2}}$$

$$I = \frac{4R^4}{8}\left(\frac{1}{2} + \frac{1}{6}\right)1 = \frac{R^4}{3} = \frac{b^4}{12}$$

201-4. Find the moment of inertia of a regular hexagon.

$$Ans.\ R = b \qquad \cos\frac{\pi}{n} = \frac{\sqrt{3}}{2} \qquad \sin\frac{\pi}{n} = \frac{1}{2} \qquad \sin\frac{2\pi}{n} = \frac{\sqrt{3}}{2}$$

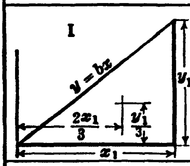
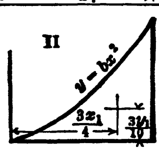
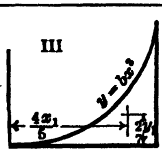
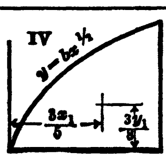
$$I = \frac{6R^4}{8}\left(\frac{3}{4} + \frac{1}{12}\right)\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{16}R^4 = 0.5412R^4 = 0.5412b^4$$

201-5. Find the moment of inertia of a regular pentagon.

Ans. $I = 0.4764R^4$; $Z_{min} = 0.4764R^3$; $Z_{max} = 0.5777R^3$.

201-6. Find the moment of inertia and the two principal section moduli for a regular octagon. Check by direct calculations without using Eq. (201.4).

TABLE 24. CENTER OF GRAVITY OF SOME PLANE AREAS

	EQUATION	AREA	M_x	$\bar{x}$	$\bar{y}$
	$y = bx$	$\frac{x_1 y_1}{2}$	$\frac{bx_1^3}{3} = \frac{x_1^2 y_1}{3}$	$\frac{2x_1}{3}$	$\frac{y_1}{3}$
	$y = bx^2$	$\frac{x_1 y_1}{3}$	$\frac{bx_1^4}{4} = \frac{x_1^2 y_1}{4}$	$\frac{3x_1}{4}$	$\frac{3y_1}{10}$
	$y = bx^3$	$\frac{x_1 y_1}{4}$	$\frac{bx_1^5}{5} = \frac{x_1^2 y_1}{5}$	$\frac{4x_1}{5}$	$\frac{4y_1}{14}$
	$y = bx^{\frac{1}{2}}$	$\frac{2x_1 y_1}{3}$	$\frac{2bx_1^{\frac{5}{2}}}{5} = \frac{2x_1^2 y_1}{5}$	$\frac{3x_1}{5}$	$\frac{3y_1}{8}$

APPENDIX

TABLE A. TENSION TEST OF STEEL, S.A.E. 1020

Carbon, 0.20; manganese, 0.52; phosphorus, 0.018; sulfur, 0.032 per cent; mean diameter, 0.752 inch; area, 0.444 square inch; gage length, 8 inches. Tested on 50,000-pound Olsen. Crosshead speed, $\frac{1}{60}$ inch per minute.

August 24, 1931

Total load, lb.	Unit stress, lb. per sq. in.	Elongation	
		In 8-in., inches	Unit, inches per inch
0	0	0	0
2,220	5,000	0.00118	0.000147
4,440	10,000	242	302
6,660	15,000	398	497
8,880	20,000	520	650
11,100	25,000	0.00660	0.000825
13,320	30,000	802	0.001002
15,540	35,000	928	1160
15,984	36,000	966	1207
16,428	37,000	996	1245
16,680	37,570	0.01010	0.001262
16,690	37,590	1102	1377
16,660	37,520	1546	1932
16,580	37,340	1866	2332
16,630	37,450	2286	2857
16,700	37,610	0.02740	0.003425
16,630	37,450	3534	4417
16,740	37,700	5182	6477
16,820	37,880	6226	7782
16,860	37,970	7362	9202
16,750	37,730	0.08562	0.010702
16,760	37,750	9302	11627
16,910	38,090	0.10482	13102
16,790	37,810	11882	14852
16,890	38,040	12902	16127
16,650	37,500	0.14872	0.018590
16,430	37,000	16042	20052
16,390	36,910	16602	20752
16,790	37,810	17096	21370
17,280	38,920	17642	22052
17,480	39,370	0.18042	0.022552
17,650	39,750	18462	23077
17,930	40,380	19526	24407
18,370	41,370	20910	26137
18,660	42,030	21950	27437
19,190	43,220	0.23990	0.029987
19,510	43,940	25486	31857
19,750	44,480	26698	33372

Total load, lb.	Unit stress, lb. per sq. in.	Elongation		
		Total, inches	Unit, inches per inch	
20,750	46,730	0.35	0.0437	
22,040	49,640	.40	.050*	
23,100	52,030	.50	.0625	
23,940	53,920	.60	.0750	
24,700	55,630	.70	.0875	
25,260	56,890	0.80	0.100	
25,620	57,700	.90	.1125	
26,030	58,630	1.00	.125	
26,210	59,030	1.10	.1375	
26,460	59,590	1.20	.150	
26,540	59,770	1.30	0.1625	
26,690	60,110	1.40	.175	
26,750	60,250	1.50	.1875	
26,770	60,290	1.60	.200	
26,770	60,290	1.70	.2125	
26,840	60,450	1.80	0.225	
26,840	60,450	1.90	.2375	Diam.
26,810	60,380	2.00	.250	neck
26,740	60,220	2.10	.2625	0.667
26,720	60,100	2.2	.275	.658
26,650	60,020	2.3	0.2875	
26,600	59,910	2.4	.300	.655
26,500	59,680	2.5	.3125	.647
26,410	59,480	2.6	.325	.637
26,040	58,650	2.7	.3375	.615
24,600	55,400	2.8	0.350	.570
22,050	49,660	2.9	.3625	.502
Ran entirely at one-sixtieth inch per minute				
21,300	47,970	2.91	.3637	.483
20,700	46,620	2.92	.3650	.476
19,370	43,630			
Broke		2.92	.3650	.435

* Machine ran at 1 in. per minute for 0.08 inch. Then at $\frac{1}{60}$ in. per minute to balance.

Interval	Top	2	3	4	5	6	7	8
Length after rupture, inches...	1.29	1.33	1.72 (neck)	1.36	1.33	1.33	1.30	1.26

TABLE B. TENSION TEST OF STEEL. S.A.E. 1045

Carbon, 0.44; manganese, 0.70; phosphorus, 0.034; sulfur, 0.023 per cent; mean diameter, 0.749 inch, area 0.441 square inch; gage length, 8 inches. Tested on 50,000-pound Olsen. Crosshead speed, $\frac{1}{60}$ inch per minute.

August 31, 1931

Total load, lb.	Unit stress, lb. per sq. in.	Elongation: In 8-in., inches	Elongation: Unit, inches per inch
0	0	0	0
441	1,000	0.00020	0.000025
882	2,000	42	52
1,323	3,000	64	80
1,764	4,000	94	0.000117
2,205	5,000	0.00120	0.000150
4,410	10,000	258	322
6,615	15,000	406	507
8,820	20,000	538	672
11,025	25,000	676	845
13,230	30,000	0.00808	0.001010
13,671	31,000	836	1045
14,112	32,000	862	1075
14,553	33,000	886	1107
14,994	34,000	916	1145
15,435	35,000	0.00924	0.001155
15,876	36,000	974	1217
16,317	37,000	0.01000	1250
16,758	38,000	1034	1292
17,100	38,780	1056	1320
17,640	40,000	0.01080	0.001350
17,840	40,450	1098	1372
18,300	41,500	1126	1407
18,860	42,770	1160	1450
19,620	44,490	1204	1505
20,850	47,280	0.01286	0.001607
21,790	49,410	1356	1695
22,260	50,480	1382	1727
23,150	52,500	1442	1802
24,070	54,580	1500	1875
24,390	55,310	0.01520	0.001900
24,710	56,030	1548	1935
24,490	55,540	1630	2037
24,620	55,830	1952	2440
24,480	55,510	2394	2992
Loose scale at top			
24,510	55,790	0.03740	0.004675
24,600	55,780	4820	6025
24,460	55,460	5434	6792
24,520	55,600	6280	7850
24,400	55,330	7520	9400
24,840	56,330	0.07774	0.009717
Scale all over			
25,200	57,140	8094	0.010117
25,580	58,000	8446	0.010557
26,530	60,160	9774	12217
26,950	61,120	0.10362	12952

Total load, lb.	Unit stress, lb. per sq. in.	Elongation: In 8-in., inches	Elongation: Unit, inches per inch
27,270	61,840	0.10846	0.013557
27,680	62,770	.11214	14017
28,200	63,950	.12186	15232
28,640	64,950	.12924	16155
29,160	66,120	.13610	17012
29,700	67,350	0.14570	0.018212
30,230	68,550	.15522	19402
30,780	69,800	.16522	20662
31,310	70,980	.17504	21880
32,180	72,970	.19066	23832
32,820	74,420	0.20320	0.025400
33,330	75,580	.21388	26735
33,960	77,010	.22906	28632
34,430	78,070	.2403C	30037
36,100	81,860	.28398	35497

Elongation taken with dividers. Total elongation 0.28 inch.

Total load*	Unit stress	Elongation: Total	Elongation: Unit	Diameter, inches
38,750	87,870	0.4	0.050	
40,420	91,660	0.5	0.0625	0.724
41,550	94,220	0.6	0.075	
42,150	97,580	0.7	0.0875	0.715
42,100	95,470	0.8	0.100	
42,650	96,710	0.9	0.1125	0.706
42,800	97,050	1.0	0.125	
42,720†	96,870	1.1	0.1375	0.696
42,930	97,350	1.21	0.1512	
42,860	97,190	1.3	0.1625	
42,600	96,600	1.41	0.1762	0.681
42,280	95,870	1.51	0.1887	0.665
41,150	93,310	1.6	0.200	0.637
38,000	86,170	....	......	0.575
37,200	84,350	Broke		
After failure		1.68	0.210	0.558

* Crosshead speed 1 inch per minute for about 0.08 inch; then at $\frac{1}{60}$ inch per minute till balanced.

† Crosshead ran too far at high speed. Stress increasing when reading was taken.

TABLE C. TENSION TEST ON HIGH-CARBON STEEL. S.A.E. 1095
Gage length, 8 inches; mean diameter, 0.7466 inch; area, 0.4378 square inch.
Tested on 100,000-pound Olsen. Crosshead speed, 0.05 inch per minute.
September 14, 1931

Unit stress, lb. per sq. in.	Unit elongation, inches per inch
0	0
1,000	0.000031
2,000	65
3,000	97
4,000	0.000127
5,000	0.000157
10,000	317
15,000	502
20,000	672
25,000	850
30,000	0.001022
35,000	1195
40,000	1372
41,000	1402
42,000	1430
43,000	0.001442
44,000	1502
45,000	1535
50,000	1725
55,000	1877
60,000	0.002047
65,000	2237
70,000	2482
75,000	3182
80,000	4554
85,000	0.006147
90,000	7937
95,000	9930
100,000	0.012275
102,790	13547
105,070	0.014840
107,350	16075
109,640	17370
111,920	18575
114,210	20127
116,490	0.021495
118,780	23067
121,060	24617
123,340	26397
124,490	27337
125,630	0.028312
126,770	29337
127,910	30367
129,050	31562
130,200	32587
131,340	0.033687
136,480	0.034907

Unit stress, lb. per sq. in.	Unit elongation, inches per inch
Elongation with dividers	
134,080	0.03625
135,910	3875
136,480	4125
138,300	425
139,220	4375
139,790	0.04625
140,250	475
141,160	4875
142,070	5125
142,760	525
143,450	0.05625
144,020	5937
144,590	6187
145,040	65
145,390	6875
145,730	0.07187
145,840	7625
145,500	8215
145,040	85
144,930	Broke

Length

Stress	Upper four divisions, in.	Lower four divisions, in.
136,480	4.14	4.14
145,500	4.33	4.32
145,040	4.34	4.34
After fracture..	4.26	4.32

Diameter readings in two planes

Division	Stress 145,500		After fracture	
Top	0.707	0.710	0.711	0.713
2	0.717	0.717	0.729	0.724
3	0.718	0.719	0.726	0.726
4	0.717	0.717	0.723	0.724
5	0.715	0.715	0.724	0.719
6	0.704	0.705	0.704	0.700
7	0.714	0.716	0.722	0.717
8	0.715	0.717	0.726	0.721
Average.	0.7152		0.7192	

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